

# INCLUSION SYSTEMS AND AMALGAMATED PRODUCTS OF PRODUCT SYSTEMS

B. V. RAJARAMA BHAT AND MITHUN MUKHERJEE

Indian Statistical Institute,  
R. V. College Post,  
Bangalore-560059, India.  
bhat@isibang.ac.in, mithun@isibang.ac.in

## Abstract<sup>12</sup>

Here we generalize the concept of Skeide product, introduced by Skeide, of two product systems via a pair of normalized units. This new notion is called amalgamated product of product systems, and now the amalgamation can be done using contractive morphisms. Index of amalgamation product (when done through units) adds up for normalized units but for non-normalized units, the index is one more than the sum. We define inclusion systems and use it as a tool for index computations. It is expected that this notion will have other uses.

## 1. INTRODUCTION

Studying quantum dynamical semigroups (or completely positive semigroups) and their dilations is important in understanding irreversible quantum dynamics. In this context, R.T. Powers posed the following problem at the 2002 AMS summer conference on ‘Advances in Quantum Dynamics’ held at Mount Holyoke: Let  $\mathcal{B}(\mathcal{H})$  and  $\mathcal{B}(\mathcal{K})$  be algebras of all bounded operators on two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ . Suppose  $\phi = \{\phi_t : t \geq 0\}$  and  $\psi = \{\psi_t : t \geq 0\}$  are two contractive completely positive (CP) semigroups on  $\mathcal{B}(\mathcal{H})$  and  $\mathcal{B}(\mathcal{K})$  respectively and  $U = \{U_t : t \geq 0\}$  and  $V = \{V_t : t \geq 0\}$  are two strongly continuous semigroups of isometries which intertwine  $\phi_t$  and  $\psi_t$  respectively. Consider the CP semigroup  $\tau_t$  on  $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$  defined by  $\tau_t \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} \phi_t(X) & U_t Y V_t^* \\ V_t Z U_t^* & \psi_t(W) \end{pmatrix}$ . How is the minimal dilation (in the sense of [4], [5]) of  $\tau$  related to minimal dilations of  $\phi$  and  $\psi$ ? In fact, Powers was interested in a more specific question. It is the following. Recall that by W. Arveson [1] we can associate a tensor product system of Hilbert spaces with every  $E$ -semigroup of  $\mathcal{B}(\mathcal{K})$ . Since the minimal dilation is unique we might say that we are associating a product system to a given contractive CP semigroup [4]. (This can also be

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<sup>1</sup>AMS Subject Classification: 46L57. Keywords: Product Systems, Completely Positive Semigroups, Inclusion Systems.

<sup>2</sup>To appear in the Journal ‘Infinite Dimensional Analysis, Quantum Probability and Related Topics’ (2010).

done more directly as in [8]). The question was to ‘What is the product system of  $\tau$  in terms of product systems of  $\phi$  and  $\psi$ . Is it the tensor product?’ This was answered by Skeide (See [7], [9], [12], [13]). It turns out that though the index of product system of  $\tau$  is sum of indices of  $\phi$  and  $\psi$ , it is not tensor product but a product introduced by Skeide in [11]. We will be calling it as Skeide product. It is a kind of amalgamated tensor product where we identify two units. Strangely proof of this fact depended upon the fact that units (intertwining semigroups)  $\{U_t\}, \{V_t\}$  are normalized, though  $\tau$  can be constructed even when they are just contractive. This raised the question as to what is the product system when the units are not normalized. We answer this question and we see that surprisingly the index increases by 1 if the units are not normalized. The original motivation in [11] was to get an appropriate product for product systems Hilbert modules where the index is additive, since there is no obvious notion of tensor products for product systems of Hilbert modules. Since this article is only about Hilbert spaces, we do not dwell more on this point.

To begin with we introduce the notion of inclusion systems. These are parametrized families of Hilbert spaces exactly like product systems except that now unitaries are replaced by isometries. Actually these objects seem to be ubiquitous in the field of product systems. Even while associating product systems to CP semigroups what one gets first are inclusion systems, and then an inductive limit procedure gives product system [8]. In [8], [6] this procedure has been elaborated and has been exploited, in the more general context of product systems of Hilbert modules. Since the same construction is being repeatedly used, it is good to extract the essence of the method and put it in a general framework. This is what is being done here. In other words we define inclusion systems and show that every inclusion system gives rise to a product system in a natural way (by taking inductive limits). It is remarkable here that basic properties of the product systems such as existence/non-existence of units, structure of morphisms etc. can be read of at the level of inclusion systems. This is the observation which we wish to stress. We believe that this technique will be very useful in many other contexts. While writing this article, we have come to know that Shalit and Solel [10], have called inclusion systems as ‘subproduct systems’ and have looked at their general theory, connections with CP semigroups, even subproduct systems of correspondences. As per authors of [10], their goal is to extend the dilation theory to more general semigroups, that is, instead of  $\mathbb{Z}_+, \mathbb{R}_+$  they wish to consider  $\mathbb{Z}_+^d, \mathbb{R}_+^d$  etc. for the time parameter. They restrict mostly to discrete semigroups as the multivariable theory is quite involved already for discrete semigroups. Consequently inductive limits do not appear in their work. So the results in [10] are quite different from ours. Since there is the possibility of confusion between ‘subproduct systems’ and ‘product subsystems’, we avoid this terminology. Occasionally we need to talk about one inclusion system contained in another one, and here too the usage ‘subproduct system’ becomes a bit awkward. Our work and some questions arose from it have motivated Tsirelson to look deeper into subproduct systems with finite dimensional spaces ([14], [15]).

In Section 3 we describe ‘amalgamation’ of two inclusion systems via a contractive morphism. This is more general than Skeide product and is useful for treating the case of more general ‘corners’ in Powers’ problem. There is an interesting relationship between units of the amalgamated product and units of the individual systems (Lemma 19). Lastly we treat the special case, where the amalgamation is done through a pair of units, using the contraction morphism  $D_t = |u_t^0\rangle\langle v_t^0|$  where  $u^0$  and  $v^0$  are two fixed units of product systems  $\mathcal{E}$  and  $\mathcal{F}$  such that  $\|u_t^0\| \|v_t^0\| \leq 1$  for all  $t > 0$ . Then it turns out that  $Ind(\mathcal{E} \otimes \mathcal{F}) = Ind(\mathcal{E}) + Ind(\mathcal{F})$  if both the units are normalized,  $Ind(\mathcal{E} \otimes_D \mathcal{F}) = Ind(\mathcal{E}) + Ind(\mathcal{F}) + 1$  otherwise.

## 2. INCLUSION SYSTEMS

**Definition 1.** An Inclusion System  $(E, \beta)$  is a family of Hilbert spaces  $E = \{E_t, t \in (0, \infty)\}$  together with isometries  $\beta_{s,t}: E_{s+t} \rightarrow E_s \otimes E_t$ , for  $s, t \in (0, \infty)$ , such that  $\forall r, s, t \in (0, \infty)$ ,  $(\beta_{r,s} \otimes 1_{E_t})\beta_{r+s,t} = (1_{E_r} \otimes \beta_{s,t})\beta_{r,s+t}$ . It is said to be a product system if further every  $\beta_{s,t}$  is a unitary.

At the moment we are not putting any measurability conditions on inclusion systems. Such technical conditions can be put when they become necessary. Of course, every product system is an inclusion system. Now here, there is a subtle point to note. Defining unitaries for product systems usually go from  $E_s \otimes E_t$  to  $E_{s+t}$  and are associative. We have taken their adjoint maps which are ‘co-associative’. So one might say that we are actually looking at ‘co-product systems’ and abusing the terminology by calling them ‘product systems’. We thank the referee for pointing this out to us.

Here are some genuine inclusion systems.

**Example 2.** Take  $E_t \equiv \mathbb{C}^2$  with orthonormal basis  $\{e_0, e_1\}$ . Define  $\beta_{s,t}: E_{s+t} \rightarrow E_s \otimes E_t$  by

$$\beta_{s,t}e_0 = e_0 \otimes e_0, \quad \beta_{s,t}(e_1) = \frac{1}{\sqrt{s+t}}(\sqrt{s}e_1 \otimes e_0 + \sqrt{t}e_0 \otimes e_1).$$

Then  $(E, \beta)$  is an inclusion system.

In the following example we are making use of concepts such as ‘units’ and spatial product systems (those which have units) as in [3]. The reader may also refer to Definition 8, below.

**Example 3.** Let  $(\mathcal{F}, C)$  be a spatial product system. Let  $\mathcal{U}^{\mathcal{F}}$  be the set of units of this product system. Now  $(E, \beta)$  with  $E_t = \overline{\text{span}}\{u_t : u \in \mathcal{U}^{\mathcal{F}}\}$ , and  $\beta_{s,t} = C_{s,t}|_{E_{s+t}}$  is an inclusion system.

Stinespring dilations of semigroups of completely positive maps is another source of inclusion systems. This will be explained towards the end of this Section.

Our first job is to show that every inclusion system leads to a product system in a natural way. Here we explain this procedure. So consider an inclusion system  $(E, \beta)$ . Let for  $t \in \mathbb{R}_+$ ,  $J_t = \{(t_n, t_{n-1}, \dots, t_1) :$

$t_i > 0, \sum_{i=1}^n t_i = t, n \geq 1\}$ . For  $\mathbf{s} = (s_m, s_{m-1}, \dots, s_1) \in J_s$ , and  $\mathbf{t} = (t_n, t_{n-1}, \dots, t_1) \in J_t$  we define  $\mathbf{s} \smile \mathbf{t} := (s_m, s_{m-1}, \dots, s_1, t_n, t_{n-1}, \dots, t_1) \in J_{s+t}$ . Now fix  $t \in \mathbb{R}_+$ . On  $J_t$  define a partial order  $\mathbf{t} \geq \mathbf{s} = (s_m, s_{m-1}, \dots, s_1)$  if for each  $i$ ,  $(1 \leq i \leq m)$  there exists (unique)  $\mathbf{s}_i \in J_{s_i}$  such that  $\mathbf{t} = \mathbf{s}_m \smile \mathbf{s}_{m-1} \smile \dots \smile \mathbf{s}_1$ . For  $\mathbf{t} = (t_n, t_{n-1}, \dots, t_1)$  in  $J_t$  define  $E_{\mathbf{t}} = E_{t_n} \otimes E_{t_{n-1}} \otimes \dots \otimes E_{t_1}$ . For  $\mathbf{s} = (s_m, \dots, s_1) \leq \mathbf{t} = (\mathbf{s}_m \smile \dots \smile \mathbf{s}_1)$  in  $J_t$ , define  $\beta_{\mathbf{t}, \mathbf{s}} : E_{\mathbf{s}} \rightarrow E_{\mathbf{t}}$  by  $\beta_{\mathbf{t}, \mathbf{s}} = \beta_{\mathbf{s}_m, s_m} \otimes \beta_{\mathbf{s}_{m-1}, s_{m-1}} \otimes \dots \otimes \beta_{\mathbf{s}_1, s_1}$  where we define  $\beta_{\mathbf{s}, s} : E_s \rightarrow E_s$  inductively as follows: Set  $\beta_{\mathbf{s}, s} = id_{E_s}$ . For  $\mathbf{s} = (s_m, s_{m-1}, \dots, s_1)$ ,  $\beta_{\mathbf{s}, s}$  is the composition of maps:

$$(\beta_{s_m, s_{m-1}} \otimes I)(\beta_{s_m + s_{m-1}, s_{m-2}} \otimes I) \dots (\beta_{s_m + \dots + s_3, s_2} \otimes I) \beta_{s_m + \dots + s_2, s_1}.$$

**Lemma 4.** *Let  $t \in \mathbb{R}_+$  be fixed and consider the partially ordered set  $J_t$  defined above. Then  $\{E_{\mathbf{t}}, \beta_{\mathbf{s}, \mathbf{r}} : \mathbf{r}, \mathbf{s}, \mathbf{t} \in J_t\}$  forms an Inductive System of Hilbert spaces in the sense that: (i)  $\beta_{\mathbf{s}, \mathbf{s}} = id_{E_s}$  for  $\mathbf{s} \in J_t$ ; (ii)  $\beta_{\mathbf{t}, \mathbf{s}} \beta_{\mathbf{s}, \mathbf{r}} = \beta_{\mathbf{t}, \mathbf{r}}$  for  $\mathbf{r} \leq \mathbf{s} \leq \mathbf{t} \in J_t$ .*

Proof: Only (ii) needs to be proved. Let  $\mathbf{r} = (r_n, \dots, r_1)$ ,  $\mathbf{s} = \mathbf{r}_n \smile \dots \smile \mathbf{r}_1$ , where  $\mathbf{r}_i = (r_{ik_i}, \dots, r_{i1})$ ,  $1 \leq i \leq n$ . So

$$\mathbf{t} = (\mathbf{r}_{nk_n} \smile \dots \smile \mathbf{r}_{n1}) \smile (\mathbf{r}_{(n-1)k_{n-1}} \smile \dots \smile \mathbf{r}_{(n-1)1}) \smile \dots \smile (\mathbf{r}_{1k_1} \smile \dots \smile \mathbf{r}_{11}).$$

Now

$$\begin{aligned} \beta_{\mathbf{t}, \mathbf{s}} \beta_{\mathbf{s}, \mathbf{r}} &= \beta_{\mathbf{t}, \mathbf{s}} (\beta_{\mathbf{r}_n, r_n} \otimes \dots \otimes \beta_{\mathbf{r}_1, r_1}) \\ &= (\beta_{\mathbf{r}_{nk_n}, r_{nk_n}} \otimes \beta_{\mathbf{r}_{nk_{n-1}}, r_{nk_{n-1}}} \otimes \dots \otimes \beta_{\mathbf{r}_{n1}, r_{n1}} \otimes \dots \otimes \beta_{\mathbf{r}_{1k_1}, r_{1k_1}} \otimes \dots \otimes \beta_{\mathbf{r}_{11}, r_{11}}) \\ &\quad (\beta_{\mathbf{r}_n, r_n} \otimes \dots \otimes \beta_{\mathbf{r}_1, r_1}) \\ &= [\beta_{(\mathbf{r}_{nk_n} \smile \dots \smile \mathbf{r}_{n1}), \mathbf{r}_n} \otimes \beta_{(\mathbf{r}_{(n-1)k_{n-1}} \smile \dots \smile \mathbf{r}_{(n-1)1}), \mathbf{r}_{n-1}} \otimes \dots \otimes \beta_{(\mathbf{r}_{1k_1} \smile \dots \smile \mathbf{r}_{11}), \mathbf{r}_1}] \\ &\quad (\beta_{\mathbf{r}_n, r_n} \otimes \dots \otimes \beta_{\mathbf{r}_1, r_1}) \\ &= \beta_{(\mathbf{r}_{nk_n} \smile \dots \smile \mathbf{r}_{n1}), \mathbf{r}_n} \otimes \beta_{(\mathbf{r}_{(n-1)k_{n-1}} \smile \dots \smile \mathbf{r}_{(n-1)1}), \mathbf{r}_{n-1}} \otimes \dots \otimes \beta_{(\mathbf{r}_{1k_1} \smile \dots \smile \mathbf{r}_{11}), \mathbf{r}_1} \\ &= \beta_{\mathbf{t}, \mathbf{r}} \end{aligned}$$

□

**Theorem 5.** *Suppose  $(E, \beta)$  is an inclusion system. Let  $\mathcal{E}_t = \text{indlim}_{J_t} E_s$  be the inductive limit of  $E_s$  over  $J_t$  for  $t > 0$ . Then  $E = \{\mathcal{E}_t : t > 0\}$  has the structure of a product system of Hilbert spaces.*

Proof: We recall four basic properties of the inductive limit construction. (i) There exist canonical isometries  $i_s : E_s \rightarrow \mathcal{E}_t$  such that given  $\mathbf{r}, \mathbf{s}$  in  $J_t$  with  $\mathbf{r} \leq \mathbf{s}$ ,  $i_s \beta_{\mathbf{s}, \mathbf{r}} = i_{\mathbf{r}}$ . (ii)  $\overline{\text{span}}\{i_s(a) : a \in E_s, \mathbf{s} \in J_t\} = \mathcal{E}_t$ . (iii) The following universal property holds: Given a Hilbert space  $\mathcal{G}$  and isometries  $g_s : E_s \rightarrow \mathcal{G}$  satisfying consistency condition  $g_s \beta_{\mathbf{s}, \mathbf{r}} = g_{\mathbf{r}}$  for all  $\mathbf{r} \leq \mathbf{s}$  there exists a unique isometry  $g : \mathcal{E}_t \rightarrow \mathcal{G}$  such that  $g_s = g i_s \forall \mathbf{s} \in J_t$ . (iv) Suppose  $K \subseteq J_t$  has the following property: Given  $\mathbf{s} \in J_t$  there exists  $\mathbf{t} \in K$

such that  $\mathbf{s} \leq \mathbf{t}$ . Then  $K$  is indeed a directed set with the order inherited from  $J_t$  and that the injection  $K \rightarrow J_t$  is a cofinal function. In other words,  $(x_{\mathbf{s}})_{\mathbf{s} \in K}$  is a subnet of  $(x_{\mathbf{t}})_{\mathbf{t} \in J_t}$ .  $\text{indlim}_{J_t} E_{\mathbf{s}} = \text{indlim}_K E_{\mathbf{s}}$ .

Define  $J_s \cup J_t = \{\mathbf{s} \cup \mathbf{t} : \mathbf{s} \in J_s, \mathbf{t} \in J_t\}$ . Given any element  $\mathbf{r} \in J_{s+t}$ , there exist  $\mathbf{s} \in J_s$  and  $\mathbf{t} \in J_t$  such that  $\mathbf{s} \cup \mathbf{t} \geq \mathbf{r}$ . So by the second property quoted above  $\mathcal{E}_{s+t} = \text{indlim}_{J_s \cup J_t} E_{\mathbf{s} \cup \mathbf{t}} = \text{indlim}_{\mathbf{s} \cup \mathbf{t} \in J_s \cup J_t} E_{\mathbf{s}} \otimes E_{\mathbf{t}}$ . Let  $i_{\mathbf{s}} : E_{\mathbf{s}} \rightarrow \mathcal{E}_s$ ,  $i_{\mathbf{t}} : E_{\mathbf{t}} \rightarrow \mathcal{E}_t$  be the canonical isometries. Consider the map  $i_{\mathbf{s}} \otimes i_{\mathbf{t}} : E_{\mathbf{s} \cup \mathbf{t}} \rightarrow \mathcal{E}_s \otimes \mathcal{E}_t$ , for  $\mathbf{s} \cup \mathbf{t} \in J_s \cup J_t$ . Note that  $\mathbf{s}' \cup \mathbf{t}' \leq \mathbf{s} \cup \mathbf{t}$  in  $J_s \cup J_t$  implies  $\mathbf{s}' \leq \mathbf{s}$ ,  $\mathbf{t}' \leq \mathbf{t}$ . Now as  $\beta_{\mathbf{s} \cup \mathbf{t}, \mathbf{s}' \cup \mathbf{t}'} = \beta_{\mathbf{s}, \mathbf{s}'} \otimes \beta_{\mathbf{t}, \mathbf{t}'}$ , we get  $(i_{\mathbf{s}} \otimes i_{\mathbf{t}}) \beta_{\mathbf{s} \cup \mathbf{t}, \mathbf{s}' \cup \mathbf{t}'} = i_{\mathbf{s}} \beta_{\mathbf{s}, \mathbf{s}'} \otimes i_{\mathbf{t}} \beta_{\mathbf{t}, \mathbf{t}'} = i_{\mathbf{s}'} \otimes i_{\mathbf{t}'}$ . So by the universal property, there is a unique isometry  $B_{s,t} : \mathcal{E}_{s+t} \rightarrow \mathcal{E}_s \otimes \mathcal{E}_t$  such that  $B_{s,t} i_{\mathbf{s} \cup \mathbf{t}} = i_{\mathbf{s}} \otimes i_{\mathbf{t}}$ . From (ii) it is clear that  $B_{s,t}$  is a unitary map from  $\mathcal{E}_{s+t}$  to  $\mathcal{E}_s \otimes \mathcal{E}_t$ .

Now to check  $(B_{r,s} \otimes 1_{\mathcal{E}_t}) B_{r+s,t} = (1_{\mathcal{E}_r} \otimes B_{s,t}) B_{r,s+t}$ , enough to check it on the vectors of the form  $i_{\mathbf{r} \cup \mathbf{s} \cup \mathbf{t}}(a \otimes b \otimes c)$ ,  $a \in E_{\mathbf{r}}$ ,  $b \in E_{\mathbf{s}}$ ,  $c \in E_{\mathbf{t}}$ . We have

$$\begin{aligned} (B_{r,s} \otimes 1_{\mathcal{E}_t}) B_{r+s,t} i_{\mathbf{r} \cup \mathbf{s} \cup \mathbf{t}}(a \otimes b \otimes c) &= (B_{r,s} \otimes 1_{\mathcal{E}_t})(i_{\mathbf{r} \cup \mathbf{s}}(a \otimes b) \otimes i_{\mathbf{t}}(c)) \\ &= B_{r,s} i_{\mathbf{r} \cup \mathbf{s}}(a \otimes b) \otimes i_{\mathbf{t}}(c) \\ &= i_{\mathbf{r}}(a) \otimes i_{\mathbf{s}}(b) \otimes i_{\mathbf{t}}(c) \end{aligned}$$

And also

$$\begin{aligned} (1_{\mathcal{E}_r} \otimes B_{s,t}) B_{r,s+t} i_{\mathbf{r} \cup \mathbf{s} \cup \mathbf{t}}(a \otimes b \otimes c) &= (1_{\mathcal{E}_r} \otimes B_{s,t})(i_{\mathbf{r}}(a) \otimes i_{\mathbf{s} \cup \mathbf{t}}(b \otimes c)) \\ &= i_{\mathbf{r}}(a) \otimes i_{\mathbf{s}}(b) \otimes i_{\mathbf{t}}(c). \end{aligned}$$

This proves the Theorem.  $\square$

**Definition 6.** Given an inclusion system  $(E, \beta)$ , the product system  $(\mathcal{E}, B)$  constructed as in the previous theorem is called the product system generated by the inclusion system  $(E, \beta)$ .

It is to be noted that if  $(E, \beta)$  is already a product system then its generated system is itself.

**Definition 7.** Let  $(E, \beta)$  and  $(F, \gamma)$  be two inclusion systems. Let  $A = \{A_t : t > 0\}$  be a family of linear maps  $A_t : E_t \rightarrow F_t$ , satisfying  $\|A_t\| \leq e^{tk}$  for some  $k \in \mathbb{R}$ . Then  $A$  is said to be a morphism or a weak morphism from  $(E, \beta)$  to  $(F, \gamma)$  if

$$A_{s+t} = \gamma_{s,t}^*(A_s \otimes A_t) \beta_{s,t} \quad \forall s, t > 0.$$

It is said to be a strong morphism if

$$\gamma_{s,t} A_{s,t} = (A_s \otimes A_t) \beta_{s,t} \quad \forall s, t > 0.$$

It is clear that every strong morphism is a weak morphism but the converse is not true. However the two notions coincide for product systems, as the linking maps are all unitaries. The exponential

boundedness condition becomes important when we take inductive limits. We also note that adjoint of a weak morphism is a weak morphism. The adjoint of a strong morphism need not be a strong morphism, but it is at least a weak morphism. Compositions of strong morphisms is a strong morphism, but this need not be true for weak morphisms.

**Definition 8.** Let  $(E, \beta)$  be an inclusion system. Let  $u = \{u_t : t > 0\}$  be a family of vectors such that (1) for all  $t > 0$ ,  $u_t \in E_t$  (2) there is a  $k \in \mathbb{R}$ , such that  $\|u_t\| \leq e^{tk}$ , for all  $t > 0$ . and (3)  $u_t \neq 0$  for some  $t > 0$ . Then  $u$  is said to be a unit or a weak unit if

$$u_{s+t} = \beta_{s,t}^*(u_s \otimes u_t) \quad \forall s, t > 0.$$

It is said to be a strong unit if

$$\beta_{s,t} u_{s+t} = u_s \otimes u_t \quad \forall s, t > 0.$$

A weak (resp. strong) unit of  $(E, \beta)$  can be thought of as a non-zero weak (resp. strong) morphism from the trivial product system  $(F, \gamma)$ , where  $F_t \equiv \mathbb{C}$  and  $\gamma_{s,t}(a) = a \otimes 1$ . As any morphism  $A : (F, \gamma) \rightarrow (E, \beta)$  is completely determined by the values  $A_t(1)$ ,  $t > 0$ . It is easy to see that  $(A_t(1))_{t>0}$  is a weak or strong unit if  $A$  is weak or strong morphism respectively.

**Lemma 9.** Let  $A_t : (E_t, \beta_{r,s}) \rightarrow (F_t, \beta'_{r,s})$  be a morphism and let  $v = (v_t)_{t>0}$  be a unit of  $F$ . Then  $(A_t^* v_t)_{t>0}$  is a unit, provided  $A_t^* v_t \neq 0$  for some  $t > 0$ .

Proof: Suppose  $\|A_t\| \leq e^{kt}$  and  $\|v_t\| \leq e^{lt}$ , for some  $l, k \in \mathbb{R}$  Now

$$\|A_t^* v_t\| \leq \|A_t^*\| \|v_t\| \leq e^{(k+l)t}.$$

So  $(A^* v)_{t>0}$  is exponentially bounded. Also

$$\begin{aligned} \beta_{s,t}^*(A_s^* v_s \otimes A_t^* v_t) &= \beta_{s,t}^*(A_s^* \otimes A_t^*)(v_s \otimes v_t) \\ &= [(A_s \otimes A_t) \beta_{s,t}]^*(v_s \otimes v_t) \\ &= [\beta'_{s,t} A_{s+t}]^*(v_s \otimes v_t) \\ &= A_{s+t}^* \beta'^*_{s,t}(v_s \otimes v_t) \\ &= A_{s+t}^* v_{s+t}. \end{aligned}$$

□

**Theorem 10.** Let  $(E, \beta)$  be an inclusion system and let  $(\mathcal{E}, B)$  be the product system generated by it. Then the canonical map  $(i_t)_{t>0} : E_t \rightarrow \mathcal{E}_t$  is an isometric strong morphism of inclusion systems. Further  $i^*$  is an isomorphism between units of  $(\mathcal{E}, B)$  and units of  $(E, \beta)$ .

Proof: The first statement is clear from the construction. For  $\mathbf{s} = \{s_n, s_{n-1}, \dots, s_1\} \in J_t$ , denote  $\mathcal{E}_{\mathbf{s}} = \mathcal{E}_{s_n} \otimes \dots \otimes \mathcal{E}_{s_1}$ . Let  $i_{\mathbf{s}} : E_{\mathbf{s}} \rightarrow \mathcal{E}_t$  be the canonical map and let  $B_{\mathbf{s},t} : \mathcal{E}_t \rightarrow \mathcal{E}_{\mathbf{s}}$  be the unitary map as defined earlier. From the proof of Theorem 5,  $B_{\mathbf{s},t} i_{\mathbf{s}} = i_{s_n} \otimes i_{s_{n-1}} \dots \otimes i_{s_1}$ . For any unit  $v$  in  $(\mathcal{E}, B)$ , denote  $v_{\mathbf{s}} = v_{s_n} \otimes \dots \otimes v_{s_1} \in \mathcal{E}_{\mathbf{s}}$ . Then from definition  $B_{\mathbf{s},t}^* v_{\mathbf{s}} = v_t$ .

Now we prove the injectivity of  $i^*$ . Consider two units  $v, w$  of  $(\mathcal{E}, B)$  such that  $i_t^* v_t = i_t^* w_t$  for all  $t > 0$ . Now for any  $\mathbf{s} \in J_t$ ,

$$\begin{aligned} i_{\mathbf{s}}^* v_t &= i_{\mathbf{s}}^* B_{\mathbf{s},t}^* v_{\mathbf{s}} \\ &= (B_{\mathbf{s},t} i_{\mathbf{s}})^* v_{\mathbf{s}} \\ &= (i_{s_n}^* \otimes \dots \otimes i_{s_1}^*) (v_{s_n} \otimes \dots \otimes v_{s_1}) \\ &= i_{s_n}^* v_{s_n} \otimes \dots \otimes i_{s_1}^* v_{s_1} \\ &= i_{s_n}^* w_{s_n} \otimes \dots \otimes i_{s_1}^* w_{s_1} \\ &= i_{\mathbf{s}}^* w_t. \end{aligned}$$

This implies  $i_{\mathbf{s}} i_{\mathbf{s}}^* v_t = i_{\mathbf{s}} i_{\mathbf{s}}^* w_t$  for all  $\mathbf{s} \in J_t$ . Note  $\mathcal{E}_t = \overline{\text{span}}\{i_{\mathbf{s}}(a) : a \in E_{\mathbf{s}}, \mathbf{s} \in J_t\}$ . The identity  $i_{\mathbf{t}} \beta_{\mathbf{t},\mathbf{s}} = i_{\mathbf{s}}$  implies that for  $\mathbf{s} \leq \mathbf{t} \in J_t$ ,

$$i_{\mathbf{s}} i_{\mathbf{s}}^* i_{\mathbf{t}} i_{\mathbf{t}}^* = i_{\mathbf{s}} i_{\mathbf{s}}^*.$$

So the net of projections  $\{i_{\mathbf{s}} i_{\mathbf{s}}^* : \mathbf{s} \in J_t\}$  converges strongly to identity. So we get  $v_t = w_t$ . From previous lemma  $i^*$  sends unit to unit, provided the image is non-trivial, which is guaranteed by injectivity of  $i^*$ .

To see surjectivity, consider a unit  $u$  of the inclusion system  $(E, \beta)$  with  $\|u_t\| \leq e^{kt}$  for some  $k \in \mathbb{R}_+$ . Fix  $t > 0$ . Define  $u_{\mathbf{s}} := u_{s_n} \otimes \dots \otimes u_{s_1}$  for  $(s_n, s_{n-1}, \dots, s_1) \in J_t$ . Now it follows easily that for  $\mathbf{s} \leq \mathbf{t}$ ,  $u_{\mathbf{s}} = \beta_{\mathbf{t},\mathbf{s}}^* u_{\mathbf{t}}$ . Now Consider the bounded net  $\{i_{\mathbf{s}} u_{\mathbf{s}}, \mathbf{s} \in J_t\}$ . For  $\mathbf{s} \leq \mathbf{t} \in J_t$ , we have

$$\begin{aligned} i_{\mathbf{s}} i_{\mathbf{s}}^* i_{\mathbf{t}} u_{\mathbf{t}} &= i_{\mathbf{s}} \beta_{\mathbf{t},\mathbf{s}}^* i_{\mathbf{t}}^* i_{\mathbf{t}} u_{\mathbf{t}} \quad (\text{as } i_{\mathbf{t}} \beta_{\mathbf{t},\mathbf{s}} = i_{\mathbf{s}}) \\ &= i_{\mathbf{s}} u_{\mathbf{s}}. \end{aligned}$$

We first claim that the bounded net  $\{i_{\mathbf{s}} u_{\mathbf{s}} : \mathbf{s} \in J_t\}$  converges to a vector  $v_t$ . For  $\mathbf{s} \leq \mathbf{t} \in J_t$ , and  $a \in \mathcal{E}_t$ ,

$$\begin{aligned} |\langle i_{\mathbf{t}} u_{\mathbf{t}} - i_{\mathbf{s}} u_{\mathbf{s}}, a \rangle| &= |\langle (I_{\mathcal{E}_t} - i_{\mathbf{s}} i_{\mathbf{s}}^*) i_{\mathbf{t}} u_{\mathbf{t}}, a \rangle| \\ &\leq \|u_{\mathbf{t}}\| \| (I_{\mathcal{E}_t} - i_{\mathbf{s}} i_{\mathbf{s}}^*) a \| \\ &\leq e^{kt} \| (I_{\mathcal{E}_t} - i_{\mathbf{s}} i_{\mathbf{s}}^*) a \|. \end{aligned}$$

As the net of projections  $\{i_{\mathbf{s}} i_{\mathbf{s}}^* : \mathbf{s} \in J_t\}$  converges strongly to identity,  $(\langle i_{\mathbf{s}} u_{\mathbf{s}}, a \rangle)_{\mathbf{s} \in J_t}$  is a Cauchy net. Set  $\phi(a) = \lim_{\mathbf{s} \in J_t} \langle i_{\mathbf{s}} u_{\mathbf{s}}, a \rangle$ . So there exists a unique vector  $v_t \in \mathcal{E}_t$  such that

$$\phi(a) = \lim_{\mathbf{s} \in J_t} \langle i_{\mathbf{s}} u_{\mathbf{s}}, a \rangle = \langle v_t, a \rangle.$$

Also

$$\langle i_{\mathbf{s}} i_{\mathbf{s}}^* v_t, a \rangle = \langle v_t, i_{\mathbf{s}} i_{\mathbf{s}}^* a \rangle = \lim_{\mathbf{t} \in J_t} \langle i_{\mathbf{t}} u_{\mathbf{t}}, i_{\mathbf{s}} i_{\mathbf{s}}^* a \rangle = \lim_{\mathbf{t} \in J_t} \langle i_{\mathbf{s}} \beta_{\mathbf{t}, \mathbf{s}}^* u_{\mathbf{t}}, a \rangle = \langle i_{\mathbf{s}} u_{\mathbf{s}}, a \rangle.$$

So we get that for any  $\mathbf{s} \in J_t$ ,

$$i_{\mathbf{s}} i_{\mathbf{s}}^* v_t = i_{\mathbf{s}} u_{\mathbf{s}}.$$

As  $\{i_{\mathbf{s}} i_{\mathbf{s}}^* : \mathbf{s} \in J_t\}$  converges strongly to identity of  $\mathcal{E}_t$ , it proves that  $(i_{\mathbf{s}} u_{\mathbf{s}})_{\mathbf{s} \in J_t}$  converges to  $v_t$  in Hilbert space norm.

Now we claim that  $(v_t)_{t \geq 0}$  is a unit of the product system  $(\mathcal{E}, B)$ . For  $a_1, a_2, \dots, a_k$  in  $E_s$  and  $b_1, b_2, \dots, b_k$  in  $E_t$  ( $k \geq 1$ ),

$$\begin{aligned} \langle B_{s,t} v_{s+t}, \sum a_i \otimes b_i \rangle &= \sum \langle v_{s+t}, B_{s,t}^* (a_i \otimes b_i) \rangle \\ &= \sum_{\mathbf{s} \sim \mathbf{t} \in J_s \sim J_t} \lim_{\mathbf{t} \in J_s \sim J_t} \langle i_{\mathbf{s} \sim \mathbf{t}} (u_{\mathbf{s}} \otimes u_{\mathbf{t}}), B_{s,t}^* (a_i \otimes b_i) \rangle \\ &= \sum_{\mathbf{s} \sim \mathbf{t} \in J_s \sim J_t} \lim_{\mathbf{t} \in J_s \sim J_t} \langle i_{\mathbf{s}} u_{\mathbf{s}} \otimes i_{\mathbf{t}} u_{\mathbf{t}}, a_i \otimes b_i \rangle \\ &= \sum_{\mathbf{s} \in J_s} \lim_{\mathbf{s} \in J_s} \langle i_{\mathbf{s}} u_{\mathbf{s}}, a_i \rangle \lim_{\mathbf{t} \in J_t} \langle i_{\mathbf{t}} u_{\mathbf{t}}, b_i \rangle \\ &= \sum \langle v_s, a_i \rangle \langle v_t, b_i \rangle \\ &= \langle v_s \otimes v_t, \sum a_i \otimes b_i \rangle. \end{aligned}$$

This proves that  $v$  is a unit. Finally, for  $a \in E_{\mathbf{t}}$ , we have

$$\begin{aligned} \langle i_{\mathbf{t}}^* v_t, a \rangle &= \langle v_t, i_{\mathbf{t}} a \rangle \\ &= \lim_{\mathbf{r}} \langle i_{\mathbf{r}} u_{\mathbf{r}}, i_{\mathbf{t}} a \rangle \\ &= \lim_{\mathbf{r}} \langle i_{\mathbf{t}}^* i_{\mathbf{r}} u_{\mathbf{r}}, a \rangle \\ &= \lim_{\mathbf{r}} \langle \beta_{\mathbf{t}, \mathbf{r}}^* i_{\mathbf{r}}^* i_{\mathbf{r}} u_{\mathbf{r}}, a \rangle \\ &= \lim_{\mathbf{r}} \langle \beta_{\mathbf{t}, \mathbf{r}}^* u_{\mathbf{r}}, a \rangle \\ &= \langle u_t, a \rangle. \end{aligned}$$

which implies  $i_{\mathbf{t}}^* v_t = u_t$ . □

By this Theorem we see that if  $u$  is a unit of an inclusion system  $(E, \beta)$  there exists a unique unit  $\hat{u}$  in  $(\mathcal{E}, B)$  such that  $i^*(\hat{u}) = u$ . We call  $\hat{u}$  as the ‘lift’ of  $u$ . It is to be noted that for two units  $u, v$  of the inclusion system,  $\langle \hat{u}_t, \hat{v}_t \rangle = \lim_{\mathbf{s} \in J_t} \langle u_{\mathbf{s}}, v_{\mathbf{s}} \rangle$ . This helps us to compute covariance functions [1] of units.

**Theorem 11.** *Let  $(E, \beta), (F, \gamma)$  be two inclusion systems generating two product systems  $(\mathcal{E}, B), (\mathcal{F}, C)$  respectively. Let  $i, j$  be the respective inclusion maps. Suppose  $A : (E, \beta) \rightarrow (F, \gamma)$  is a weak morphism then there exists a unique morphism  $\hat{A} : (\mathcal{E}, B) \rightarrow (\mathcal{F}, C)$  such that  $A_s = j_s^* \hat{A}_s i_s$  for all  $s$ . This is a one to one correspondence of weak morphisms. Further more,  $\hat{A}$  is isometric/unitary if  $A$  is isometric/unitary.*



Proof: If  $\hat{A}$  is a morphism of product systems then  $\{A_s = j_s^* \hat{A}_s i_s\}_{s>0}$  is clearly a weak morphism of inclusion systems. Conversely suppose  $A : (E, \beta) \rightarrow (F, \gamma)$  is a morphism with  $\|A_t\| \leq e^{kt}$  for some  $k > 0$ . Define  $A_s : E_s \rightarrow F_s$  by  $A_s = A_{s_1} \otimes \cdots \otimes A_{s_n}$ . Let  $i_s : E_s \rightarrow \mathcal{E}_s$  and  $j_s : F_s \rightarrow \mathcal{F}_s$  be the canonical maps. The hypothesis implies that for  $\mathbf{s} \leq \mathbf{t}$ ,

$$\gamma_{\mathbf{s}, \mathbf{t}}^* A_{\mathbf{t}} \beta_{\mathbf{s}, \mathbf{t}} = A_{\mathbf{s}}.$$

Consider for  $\mathbf{s} \in J_s$ ,  $\Phi_{\mathbf{s}} = j_{\mathbf{s}} A_{\mathbf{s}} i_{\mathbf{s}}^*$ . Set  $P_{\mathbf{r}} = j_{\mathbf{r}} j_{\mathbf{r}}^*$  and  $Q_{\mathbf{r}} = i_{\mathbf{r}} i_{\mathbf{r}}^*$ . A simple computation shows that for  $\mathbf{r} \leq \mathbf{s}$ ,

$$P_{\mathbf{r}} \Phi_{\mathbf{s}} Q_{\mathbf{r}} = \Phi_{\mathbf{r}}.$$

For  $\mathbf{s} \leq \mathbf{t} \in J_t$ ,  $a \in \mathcal{E}_t$  and  $b \in \mathcal{F}_t$

$$\begin{aligned} |\langle (\Phi_{\mathbf{t}} - \Phi_{\mathbf{s}})a, b \rangle| &= |\langle (\Phi_{\mathbf{t}} - P_{\mathbf{s}} \Phi_{\mathbf{t}} Q_{\mathbf{s}})a, b \rangle| \\ &\leq |\langle (\Phi_{\mathbf{t}} - \Phi_{\mathbf{t}} Q_{\mathbf{s}})a, b \rangle| + |\langle (\Phi_{\mathbf{t}} Q_{\mathbf{s}} - P_{\mathbf{s}} \Phi_{\mathbf{t}} Q_{\mathbf{s}})a, b \rangle| \\ &= |\langle (I_{\mathcal{F}_t} - Q_{\mathbf{s}})a, \Phi_{\mathbf{t}}^* b \rangle| + |\langle \Phi_{\mathbf{t}} Q_{\mathbf{s}} a, (I_{\mathcal{E}_t} - P_{\mathbf{s}})b \rangle| \\ &\leq e^{kt} \|(I_{\mathcal{F}_t} - Q_{\mathbf{s}})a\| \|b\| + e^{kt} \|a\| \|b\|. \end{aligned}$$

Imitating the proof in the Theorem 10,  $(\Phi_{\mathbf{s}})_{\mathbf{s} \in J_t}$  has a weak limit say  $\hat{A}_s$ . Now for  $\mathbf{s} \in J_s$ , we get

$$\begin{aligned} \langle j_{\mathbf{s}}^* \hat{A}_s i_{\mathbf{s}} a, b \rangle &= \langle \hat{A}_s i_{\mathbf{s}} a, j_{\mathbf{s}} b \rangle \\ &= \lim \langle \Phi_{\mathbf{r}} i_{\mathbf{s}} a, j_{\mathbf{s}} b \rangle \\ &= \lim \langle j_{\mathbf{s}}^* j_{\mathbf{r}} A_{\mathbf{r}} i_{\mathbf{r}}^* i_{\mathbf{t}} a, b \rangle \\ &= \lim \langle \gamma_{\mathbf{s}, \mathbf{r}}^* A_{\mathbf{r}} \beta_{\mathbf{s}, \mathbf{r}} a, b \rangle \\ &= \langle A_{\mathbf{s}} a, b \rangle. \end{aligned}$$

This implies that  $A_s = j_s^* \hat{A}_s i_s$  and in particular  $A_s = j_s^* \hat{A}_s i_s$ . Now we claim that  $\hat{A}_s$  is a morphism of product systems. For any  $\mathbf{s} \in J_s$ ,  $\mathbf{t} \in J_t$ ,  $a \in E_s$ ,  $b \in E_t$ ,  $c \in F_s$ ,  $d \in F_t$  we have

$$\begin{aligned} \langle C_{s,t}^* (\hat{A}_s \otimes \hat{A}_t) B_{s,t} i_{\mathbf{s} \smile \mathbf{t}}(a \otimes b), j_{\mathbf{s} \smile \mathbf{t}}(c \otimes d) \rangle &= \langle (\hat{A}_s \otimes \hat{A}_t)(i_{\mathbf{s}} \otimes i_{\mathbf{t}})(a \otimes b), j_{\mathbf{s}} \otimes j_{\mathbf{t}}(c \otimes d) \rangle \\ &= \langle j_{\mathbf{s}}^* \hat{A}_s i_{\mathbf{s}} \otimes j_{\mathbf{t}}^* \hat{A}_t i_{\mathbf{t}}(a \otimes b), (c \otimes d) \rangle \\ &= \langle (A_{\mathbf{s}} \otimes A_{\mathbf{t}})(a \otimes b), (c \otimes d) \rangle \\ &= \langle A_{\mathbf{s} \smile \mathbf{t}}(a \otimes b), (c \otimes d) \rangle \\ &= \langle j_{\mathbf{s} \smile \mathbf{t}}^* \hat{A}_{s+t} i_{\mathbf{s} \smile \mathbf{t}}(a \otimes b), (c \otimes d) \rangle \\ &= \langle \hat{A}_{s+t} i_{\mathbf{s} \smile \mathbf{t}}(a \otimes b), j_{\mathbf{s} \smile \mathbf{t}}(c \otimes d) \rangle. \end{aligned}$$

The one to one property can be proved imitating the proof in Theorem 10. The second statement is obvious.  $\square$

As a special case we have the following universal property for (weak/strong) morphisms.

**Corollary 12.** *Let  $(\mathcal{E}, B)$  be a product system generated by an inclusion system  $(E, \beta)$  with canonical map  $i$ . Suppose  $(\mathcal{F}, C)$  is a product system with isometric morphisms of inclusion system  $m : E \rightarrow \mathcal{F}$ . Then there exists unique isometric morphism of product system  $\hat{m} : \mathcal{E} \rightarrow \mathcal{F}$  such that  $\hat{m}_s i_s = m_s$  for all  $s > 0$ .*

With basic theory of inclusion systems and their morphisms in place, we look at inclusion systems arising from quantum dynamical semigroups. Though this is part of folklore, as we are going to need it in the next Section we put in some details. Let  $H$  be a Hilbert space and let  $\mathcal{B}(H)$  be the algebra of all bounded operators on  $H$ . Let  $\tau = \{\tau_t : t \geq 0\}$  be a quantum dynamical semigroup on  $\mathcal{B}(H)$ , that is, a one parameter semigroup of normal, contractive, completely positive maps of  $\mathcal{B}(H)$ . For  $t \geq 0$ , let  $(\pi_t, V_t, K_t)$  be a Stinespring dilation of  $\tau_t$ :  $K_t$  is a Hilbert space,  $V_t \in \mathcal{B}(H, K_t)$ , and  $\pi_t$  is a normal representation of  $\mathcal{B}(H)$  on  $K_t$ , such that,

$$\tau_t(X) = V_t^* \pi_t(X) V_t \quad \forall X \in \mathcal{B}(H).$$

We will not need minimality ( $K_t = \overline{\text{span}}\{\pi_t(X)V_t h : X \in \mathcal{B}(H), h \in H\}$ ). Now fix a unit vector  $a \in H$ , take

$$E_t = \overline{\text{span}}\{\pi_t(|a\rangle\langle g|)V_t h : g, h \in H\} \subseteq K_t.$$

Up to unitary equivalence the Hilbert space  $E_t$  does not depend upon the Stinespring dilation or the choice of the reference vector  $a$ . For any two unit vectors  $a$  and  $a'$ , The map

$$\pi_t(|a\rangle\langle g|)V_t h \rightarrow \pi_t(|a'\rangle\langle g|)V_t h$$

extends as a unitary between  $E_t^a$  and  $E_t^{a'}$  as

$$\langle \pi_t(|a\rangle\langle g_1|)V_t h_1, \pi_t(|a\rangle\langle g_2|)V_t h_2 \rangle = \langle h_1, \tau_t(|g_1\rangle\langle g_2|)h_2 \rangle = \langle \pi_t(|a'\rangle\langle g_1|)V_t h_1, \pi_t(|a'\rangle\langle g_2|)V_t h_2 \rangle$$

We may also construct  $E_t$  more abstractly by the usual quotienting and completing procedure on defining

$$\langle g_1 \otimes h_1, g_2 \otimes h_2 \rangle_{\tau_t} := \langle h_1, \tau_t(|g_1\rangle\langle g_2|)h_2 \rangle,$$

for  $g_1 \otimes h_1, g_2 \otimes h_2 \in H^* \otimes H$ . Now fix an ortho-normal basis  $\{e_k\}$  of  $H$  and define  $\beta_{s,t} : E_{s+t} \rightarrow E_s \otimes E_t$  by

$$\beta_{s,t}(\pi_{s+t}(|a\rangle\langle g|)V_{s+t}h) = \sum_k \pi_s(|a\rangle\langle g|)V_s e_k \otimes \pi_t(|a\rangle\langle e_k|)V_t h.$$

Then  $(E, \beta)$  is an inclusion system. Indeed,

$$\begin{aligned}
& \left\langle \sum_k \pi_s(|a\rangle\langle g_1|) V_s e_k \otimes \pi_t(|a\rangle\langle e_k|) V_t h_1, \sum_l \pi_s(|a\rangle\langle g_2|) V_s e_l \otimes \pi_t(|a\rangle\langle e_l|) V_t h_2 \right\rangle \\
&= \sum_{k,l} \langle e_k, \tau_s(|g_1\rangle\langle g_2|) e_l \rangle \cdot \langle h_1, \tau_t(|e_k\rangle\langle e_l|) h_2 \rangle \\
&= \langle h_1, \tau_t(| \sum_{k,l} \langle e_k, \tau_s(|g_1\rangle\langle g_2|) e_l \rangle e_k \rangle \langle e_l|) h_2 \rangle \\
&= \langle h_1, \tau_t(| \sum_l \tau_s(|g_1\rangle\langle g_2|) e_l \rangle \langle e_l|) h_2 \rangle \\
&= \langle h_1, \tau_t(\tau_s(|g_1\rangle\langle g_2|) \sum_{l=1}^{\infty} (|e_l\rangle\langle e_l|)) h_2 \rangle \\
&= \langle h_1, \tau_t(\tau_s(|g_1\rangle\langle g_2|)) h_2 \rangle \\
&= \langle h_1, \tau_{s+t}(|g_1\rangle\langle g_2|) h_2 \rangle.
\end{aligned}$$

So  $\beta_{s,t}$  is an isometry, and the associativity property can also be verified by direct computation: For  $r, s, t > 0$ ,

$$\begin{aligned}
& (I_r \otimes \beta_{s+t}) \beta_{r,s+t} (\pi_{r+s+t}(|a\rangle\langle g|) V_{r+s+t} g) \\
&= (I_r \otimes \beta_{s+t}) \sum_k [\pi_r(|a\rangle\langle g|) V_r e_k \otimes \pi_{s+t}(|a\rangle\langle e_k|) V_{s+t} g] \\
&= \sum_{k,l} \pi_r(|a\rangle\langle g|) V_r e_k \otimes \pi_s(|a\rangle\langle e_k|) V_s e_l \otimes \pi_t(|a\rangle\langle e_l|) V_t g \\
&= (\beta_{r,s} \otimes I_t) \cdot (\pi_{r+s}(|a\rangle\langle g|) V_{r+s} e_l \otimes \pi_t(|a\rangle\langle e_l|) V_t g) \\
&= (\beta_{r,s} \otimes I_t) \beta_{r+s,t} (\pi_{r+s+t}(|a\rangle\langle g|) V_{r+s+t} g).
\end{aligned}$$

Now we will show that  $\beta$  does not depend upon the choice of the orthonormal basis. Let  $e = (e_i)_{i=1}^{\infty}$  and  $f = (f_j)_{j=1}^{\infty}$  be two orthonormal bases of the Hilbert space  $H$ . Now denoting the associated  $\beta$  maps by  $\beta^e, \beta^f$  respectively, we get

$$\begin{aligned}
& \langle \beta_{s,t}^e \pi_{s+t}(|a\rangle\langle g|) V_{s+t} h, \beta_{s,t}^f \pi_{s+t}(|a\rangle\langle g|) V_{s+t} h \rangle \\
&= \sum_{i,j} \langle \pi_s(|a\rangle\langle g|) V_s e_i \otimes \pi_t(|a\rangle\langle e_i|) V_t h, \pi_s(|a\rangle\langle g|) V_s f_j \otimes \pi_t(|a\rangle\langle f_j|) V_t h \rangle \\
&= \sum_{i,j} \langle e_i, \tau_s(|g\rangle\langle g|) f_j \rangle \cdot \langle h, \tau_t(|e_i\rangle\langle f_j|) h \rangle \\
&= \langle h, \tau_t(| \sum_{i,j} \langle e_i, \tau_s(|g\rangle\langle g|) f_j \rangle e_i \rangle \langle f_j|) h \rangle \\
&= \langle h, \tau_t(\tau_s(|g\rangle\langle g|) \sum_{j=1}^{\infty} (|f_j\rangle\langle f_j|)) h \rangle \\
&= \langle h, \tau_{s+t}(|g\rangle\langle g|) h \rangle.
\end{aligned}$$

So

$$\begin{aligned}
& \|\beta_{s,t}^e \pi_{s+t}(|a\rangle\langle g|)V_{s+t}h - \beta_{s,t}^f \pi_{s+t}(|a\rangle\langle g|)V_{s+t}h\|^2 \\
&= \langle h, \tau_{s+t}(|g\rangle\langle g|)h \rangle - \langle h, \tau_{s+t}(|g\rangle\langle g|)h \rangle \\
&\quad - \langle h, \tau_{s+t}(|g\rangle\langle g|)h \rangle + \langle h, \tau_{s+t}(|g\rangle\langle g|)h \rangle \\
&= 0
\end{aligned}$$

Now we recall the dilation theorem for quantum dynamical semigroups (This was proved in [4] for unital quantum dynamical semigroups and was extended to the non-unital case in [5]): Given a quantum dynamical semigroup  $\tau$  on  $\mathcal{B}(\mathcal{H})$  there exists a pair  $(\theta, \mathcal{K})$  where  $\mathcal{K}$  is a Hilbert space containing  $\mathcal{H}$  and  $\theta$  is an  $E$ -semigroup of  $\mathcal{B}(\mathcal{K})$  such that,

$$\tau_t(X) = P\theta_t(X)P \quad \forall X \in \mathcal{B}(\mathcal{H}), t \geq 0,$$

where  $P$  is the projection of  $\mathcal{K}$  onto  $\mathcal{H}$  and  $X \in \mathcal{B}(\mathcal{H})$  is identified with  $PXP$  in  $\mathcal{B}(\mathcal{K})$ . Furthermore, we can choose  $\mathcal{K}$  such that,

$$\overline{\text{span}}\{\theta_{r_1}(X_1) \dots \theta_{r_n}(X_n)h : r_1 \geq r_2 \geq \dots \geq r_n \geq 0, X_1, \dots, X_n \in \mathcal{B}(\mathcal{H}), h \in \mathcal{H}, n \geq 0\} = \mathcal{K}.$$

Such a pair  $(\theta, \mathcal{K})$  is unique up to unitary equivalence and is called the minimal dilation of  $\tau$ . The minimal dilation  $\theta$  is unital if and only if  $\tau$  is unital. In the following, we need another basic property of minimal dilation: The vector  $\theta_{r_1}(X_1) \dots \theta_{r_n}(X_n)h$  appearing above remains unchanged if we drop any  $\theta_{r_k}(X_k)$  from the expression, if  $X_k = 1_{\mathcal{H}}$ . In the unital case, this fact follows easily from the property that  $P = 1_{\mathcal{H}}$  is an increasing projection for  $\theta$ . It is a bit more involved in the non-unital case.

**Theorem 13.** *Let  $\tau$  be a quantum dynamical semigroup of  $\mathcal{B}(H)$  and let  $(E, \beta)$  be the associated inclusion system defined above. Let  $\theta$  acting on  $\mathcal{B}(K)$  (with  $H \subset K$ ) be the minimal  $E$ -semigroup dilation of  $\tau$ . Let  $(\mathcal{F}, C)$  be the inclusion system of  $\theta$ , considered as a quantum dynamical semigroup. Then  $(\mathcal{F}, C)$  is a product system and is isomorphic to the product system  $(\mathcal{E}, B)$  generated by  $(E, \beta)$ .*

Proof: Since for every  $t$ ,  $\theta_t$  is a  $*$ -endomorphism, its minimal Stinespring dilation is itself. Then it is easily seen that  $F_t = \overline{\text{span}}\{\theta_t(|a\rangle\langle x|)y : x, y \in K\}$  and  $C_{s,t} : \mathcal{F}_{s+t} \rightarrow \mathcal{F}_s \otimes \mathcal{F}_t$ , defined by

$$C_{s,t}(\theta_{s+t}(|a\rangle\langle x|)y) = \sum_k \theta_s(|a\rangle\langle x|)e_k \otimes \theta_t(|a\rangle\langle e_k|)y,$$

is a unitary with

$$C_{s,t}^*(\theta_s(|a\rangle\langle x_1|)y_1 \otimes \theta_t(|a\rangle\langle x_2|)y_2) = \theta_{s+t}(|a\rangle\langle x_1|)\theta_t(|y_1\rangle\langle x_2|)y_2.$$

Therefore  $(\mathcal{F}, C)$  is a product system.

Define  $m_s : E_s \rightarrow \mathcal{F}_s$  by  $m_s(\pi_s(|a\rangle\langle g|)V_s h) = \theta_s(|a\rangle\langle g|)h$  where  $a$  is a unit vector in  $\mathcal{H}$ . Clearly  $m_s$  is a linear isometry. We see that  $m$  is a strong morphism of inclusion systems:

$$\begin{aligned}
& C_{s,t}^*(m_s \otimes m_t)\beta_{s,t}(\pi_{s+t}(|a\rangle\langle g|)V_{s+t}h) \\
&= C_{s,t}^*(m_s \otimes m_t)(\sum_k \pi_s(|a\rangle\langle g|V_s e_k \otimes \pi_t(|a\rangle\langle e_k|)V_t h)) \\
&= C_{s,t}^*(\sum_k \theta_s(|a\rangle\langle g|)e_k \otimes \theta_t(|a\rangle\langle e_k|)h) \\
&= \sum_k \theta_{s+t}(|a\rangle\langle g|)\theta_t(|e_k\rangle\langle e_k|)h \\
&= \theta_{s+t}(|a\rangle\langle g|)\theta_t(1_{\mathcal{H}})h \\
&= \theta_{s+t}(|a\rangle\langle g|)h \\
&= m_{s+t}(\pi_{s+t}(|a\rangle\langle g|)V_{s+t}h).
\end{aligned}$$

Now by Corollary 12, there exists an isometric morphism  $\hat{m} : (\mathcal{E}, B) \rightarrow (\mathcal{F}, C)$  satisfying  $\hat{m}_s i_s = m_s$  where  $i : (E, \beta) \rightarrow (\mathcal{E}, B)$  is the natural inclusion map. Unitarity of  $\hat{m}$  follows easily from the minimality of the dilation.  $\square$

This theorem provides us with a plenty of inclusion systems with finite dimensional systems, as we can consider contractive CP semigroups on  $\mathcal{B}(H)$  with  $\dim H < \infty$ . For instance, one gets the inclusion system of Example 2, by considering the following CP semigroup on  $\mathcal{B}(\mathbb{C}^2)$  :

$$T_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{-\alpha t} \begin{pmatrix} a + td & b \\ c & d \end{pmatrix},$$

where  $\alpha$  is a suitable positive real number so as to make the semigroup contractive (This semigroup is not unital, but that does not matter). However, all such inclusion systems would only generate type I product systems, as it is well-known that the associated product systems of CP semigroups with bounded generators are always type I. This raises the natural question as to whether inclusion systems  $(E, \beta)$ , where  $\dim(E_t) \leq N$  for some natural number  $N$  always generate type I product systems (One has to be a bit cautious here as the product system generated may contain non-separable Hilbert spaces). Recently this has been answered in the affirmative by B. Tsirelson for the case  $N = 2$ . (See [14], [15]).

**Proposition 14.** *Let  $(\mathcal{F}, C)$  be a spatial product system and let  $(E, \beta)$  be the inclusion system formed by the linear spans of units (See Example 2). Then the product system  $(\mathcal{E}, B)$  generated by  $(E, \beta)$  is the type I part of  $(\mathcal{F}, C)$ .*

Proof: This is obvious, as the space  $E_{\mathbf{t}} = E_{t_n} \otimes \cdots \otimes E_{t_1}$  can be identified with  $\overline{\text{span}}\{u_{t_n}^n \otimes \cdots \otimes u_{t_1}^1 : u^1, \dots, u^n \in \mathcal{U}^{\mathcal{E}}\}$ .

$\square$

## 3. AMALGAMATION

Suppose  $H$  and  $K$  are two Hilbert spaces and  $D : K \rightarrow H$  is a linear contraction. Define a semi inner product on  $H \oplus K$  by

$$\begin{aligned} \left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_D &= \langle u_1, u_2 \rangle + \langle u_1, Dv_2 \rangle + \langle Dv_1, u_2 \rangle + \langle v_1, v_2 \rangle \\ &= \left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \tilde{D} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle, \end{aligned}$$

where  $\tilde{D} := \begin{bmatrix} I & D \\ D^* & I \end{bmatrix}$ . Note that as  $D$  is contractive,  $\tilde{D}$  is positive definite. Take

$$N = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_D = 0 \right\}.$$

Then  $N$  is the kernel of bounded operator  $\tilde{D}$  and hence it is a closed subspace of  $H \oplus K$ . Set  $G$  as completion of  $(H \oplus K)/N$  with respect to norm of  $\langle \cdot, \cdot \rangle_D$ . We denote  $G$  by  $H \oplus_D K$  and further denote the image of vector  $\begin{pmatrix} u \\ v \end{pmatrix}$  by  $\begin{bmatrix} u \\ v \end{bmatrix}$  for  $u \in H$  and  $v \in K$ . Now

$$\left\langle \begin{bmatrix} u_1 \\ 0 \end{bmatrix}, \begin{bmatrix} u_2 \\ 0 \end{bmatrix} \right\rangle_D = \langle u_1, u_2 \rangle_H; \quad \left\langle \begin{bmatrix} 0 \\ v_1 \end{bmatrix}, \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \right\rangle_D = \langle v_1, v_2 \rangle_K.$$

So  $H$  and  $K$  are naturally embedded in  $H \oplus_D K$  and their closed linear span is  $H \oplus_D K$  but they need not be orthogonal. We call  $H \oplus_D K$  as the amalgamation of  $H$  and  $K$  through  $D$ . It is to be noted that if range  $(\tilde{D})$  is closed, then no completion is needed in the construction, and every vector of  $G$  is of the form  $\begin{bmatrix} u \\ v \end{bmatrix}$  for  $u \in H$  and  $v \in K$ .

In the converse direction, if  $H$  and  $K$  are two closed subspaces of a Hilbert space  $G$ . Then by a simple application of Riesz representation theorem, there exists unique contraction  $D : K \rightarrow H$  such that for  $u \in H, v \in K$

$$\langle u, v \rangle_G = \langle u, Dv \rangle.$$

Now we consider amalgamation at the level of inclusion systems. Let  $(E, \beta)$  and  $(F, \gamma)$  be two inclusion systems. Let  $D = \{D_s : s > 0\}$  be a weak contractive morphism from  $F$  to  $E$ . Define  $G_s := E_s \oplus_{D_s} F_s$  and  $\delta_{s,t} := i_{s,t}(\beta_{s,t} \oplus_D \gamma_{s,t})$  where  $i_{s,t} : (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t) \rightarrow G_s \otimes G_t$  is the map defined by

$$i_{s,t} \begin{bmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_2 \end{bmatrix},$$

and  $(\beta_{s,t} \oplus_D \gamma_{s,t}) : E_{s+t} \oplus_{D_{s+t}} F_{s+t} \rightarrow E_s \otimes E_t \oplus_{D_s \otimes D_t} F_s \otimes F_t$  is the map defined by

$$(\beta_{s,t} \oplus_D \gamma_{s,t}) \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \beta_{s,t}(u) \\ \gamma_{s,t}(v) \end{bmatrix}$$

**Lemma 15.** *The maps  $i_{s,t} : (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t) \rightarrow G_s \otimes G_t$  and  $(\beta_{s,t} \oplus_D \gamma_{s,t}) : E_{s+t} \oplus_{D_{s+t}} F_{s+t} \rightarrow (E_s \otimes E_t) \oplus_{D_s \otimes D_t} (F_s \otimes F_t)$  are well defined isometries.*

Proof: It is enough to check that the maps are inner product preserving on elementary tensors. Observe that

$$\begin{aligned} & \left\langle i_{s,t} \begin{bmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{bmatrix}, i_{s,t} \begin{bmatrix} u'_1 \otimes u'_2 \\ v'_1 \otimes v'_2 \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_2 \end{bmatrix}, \begin{bmatrix} u'_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u'_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v'_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v'_2 \end{bmatrix} \right\rangle \\ &= \langle u_1, u'_1 \rangle \langle u_2, u'_2 \rangle + \langle v_1, v'_1 \rangle \langle v_2, v'_2 \rangle + \langle u_1, D_s v'_1 \rangle \langle u_2, D_t v'_2 \rangle + \langle D_s v_1, u'_1 \rangle \langle D_t v_2, u'_2 \rangle \\ &= \left\langle \begin{bmatrix} u_1 \otimes u_2 \\ v_1 \otimes v_2 \end{bmatrix}, \begin{bmatrix} u'_1 \otimes u'_2 \\ v'_1 \otimes v'_2 \end{bmatrix} \right\rangle. \end{aligned}$$

and

$$\begin{aligned} & \left\langle (\beta_{s,t} \oplus_D \gamma_{s,t}) \begin{bmatrix} u \\ v \end{bmatrix}, (\beta_{s,t} \oplus_D \gamma_{s,t}) \begin{bmatrix} u' \\ v' \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \beta_{s,t}(u) \\ \gamma_{s,t}(v) \end{bmatrix}, \begin{bmatrix} \beta_{s,t}(u') \\ \gamma_{s,t}(v') \end{bmatrix} \right\rangle \\ &= \langle \beta_{s,t}(u), \beta_{s,t}(u') \rangle + \langle \gamma_{s,t}(v), \gamma_{s,t}(v') \rangle \\ &\quad + \langle \beta_{s,t}(u), (D_s \otimes D_t) \gamma_{s,t}(v') \rangle + \langle (D_s \otimes D_t) \gamma_{s,t}(v), \beta_{s,t}(u') \rangle \\ &= \langle u, u' \rangle + \langle v, v' \rangle + \langle u, D_{s+t} v' \rangle + \langle D_{s+t} v, u' \rangle \\ &= \left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u' \\ v' \end{bmatrix} \right\rangle. \end{aligned}$$

□

**Proposition 16.** *Let  $(G, \delta) = \{G_s, \delta_{s,t} : s, t > 0\}$  be defined as above. Then  $\{G, \delta\}$  forms an inclusion system.*

Proof: Being composition of two isometries,  $\delta_{s,t}$  is an isometry. Define  $i_{r,s,t} : (E_r \otimes E_s \otimes E_t) \oplus_{D_r \otimes D_s \otimes D_t} (F_r \otimes F_s \otimes F_t) \rightarrow G_r \otimes G_s \otimes G_t$  by

$$i_{r,s,t} \begin{bmatrix} u_1 \otimes u_2 \otimes u_3 \\ v_1 \otimes v_2 \otimes v_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u_2 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} u_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ v_3 \end{bmatrix}$$

It can be shown similarly that  $i_{r,s,t}$  is an isometry. For  $r, s, t \in \mathbb{R}_+$

$$\begin{aligned} (\delta_{r,s} \otimes 1_{G_t})\delta_{r+s,t} &= [i_{r,s}(\beta_{r,s} \oplus_D \gamma_{r,s}) \otimes 1_{G_t}]i_{r+s,t}(\beta_{r+s,t} \oplus_D \gamma_{r+s,t}) \\ &= i_{r,s,t}((\beta_{r,s} \otimes 1_{E_t})\beta_{r+s,t} \oplus_D (\gamma_{r,s} \otimes 1_{E_t})\gamma_{r+s,t}) \end{aligned}$$

Similarly

$$(1_{G_r} \otimes \delta_{s,t})\delta_{r,s+t} = i_{r,s,t}((1_{E_r} \otimes \beta_{s,t})\beta_{r,s+t} \oplus_D (1_{F_r} \otimes \gamma_{s,t})\gamma_{r,s+t})$$

As  $(E, \beta)$ ,  $(F, \gamma)$  are inclusion systems, so is  $(G, \delta)$ .  $\square$

**Definition 17.** The inclusion system  $(G, \delta)$  constructed above is called the amalgamation of inclusion systems  $(E, \beta)$  and  $(F, \gamma)$  via the morphism  $D$ . If  $(\mathcal{E}, B)$ ,  $(\mathcal{F}, C)$ , and  $(\mathcal{G}, L)$  are product systems generated respectively by  $(E, \beta)$ ,  $(F, \gamma)$ , and  $(G, \delta)$ , then  $(\mathcal{G}, L)$  is said to be the amalgamated product of  $(\mathcal{E}, B)$  and  $(\mathcal{F}, C)$  via  $D$  and is denoted by  $\mathcal{G} =: \mathcal{E} \otimes_D \mathcal{F}$ .

In this Definition notice that as  $D$  is a weak morphism of inclusion system we will get a lift  $\hat{D} : \mathcal{F} \rightarrow \mathcal{E}$ . We can also define  $\mathcal{E} \otimes_{\hat{D}} \mathcal{F}$ . It can be seen easily that product system generated by the amalgamated product of  $(E, \beta)$  and  $(F, \gamma)$  via  $D$  is same as  $\mathcal{E} \otimes_{\hat{D}} \mathcal{F}$ , so that the definition of amalgamated product is unambiguous. This is true because of the following universal property of amalgamation.

**Proposition 18.** Let  $(G, \delta)$  be the amalgamated inclusion system of two inclusion systems  $(E, \beta)$ ,  $(F, \gamma)$  via a morphism  $D$  from  $F$  to  $E$ . Let  $(\mathcal{G}, L)$  be the product system generated by  $(G, \delta)$ . Given any inclusion system  $(H, \eta)$  with weak isometric morphisms  $i : (E, \beta) \rightarrow (H, \eta)$  and  $j : (F, \gamma) \rightarrow (H, \eta)$  with  $\langle i_t a, j_t b \rangle = \langle a, D_t b \rangle$  for all  $a \in E_t$  and  $b \in F_t$ , there exists unique isometric morphism of product system  $\hat{A} : (\mathcal{G}, L) \rightarrow (\mathcal{H}, W)$  such that  $l_t^* \hat{A}_t (k_t \begin{bmatrix} a \\ b \end{bmatrix}) = (i_t(a) + j_t(b))$ , where  $(\mathcal{H}, W)$  is the product system generated by  $(H, \eta)$ , and  $k_t : G_t \rightarrow \mathcal{G}_t$ ,  $l_t : H_t \rightarrow \mathcal{H}_t$  are respective canonical maps.

Proof: Define  $A_t : G_t \rightarrow \mathcal{H}_t$  by  $A_t \begin{bmatrix} a \\ b \end{bmatrix} = i_t(a) + j_t(b)$ . Clearly  $A_t$  is an isometry for each  $t$ . Now  $W_{s,t}^*(A_s \otimes A_t)\delta_{s,t} \begin{bmatrix} a \\ b \end{bmatrix} = W_{s,t}^*(A_s \otimes A_t)i_{s,t} \begin{bmatrix} \beta_{s,t}(a) \\ \gamma_{s,t}(b) \end{bmatrix} = W_{s,t}^*(i_s \otimes i_t)\beta_{s,t}(a) + W_{s,t}^*(j_s \otimes j_t)\gamma_{s,t}(b) = i_{s+t}(a) + j_{s+t}(b) = A_{s+t} \begin{bmatrix} a \\ b \end{bmatrix}$ . This means that  $A$  is a weak isometric morphism of inclusion systems. So

it lifts to an isometric morphism of product systems  $\hat{A} : \mathcal{G} \rightarrow \mathcal{H}_t$  such that  $l_t^* \hat{A}_t k_t \begin{bmatrix} a \\ b \end{bmatrix} = (i_t(a) + j_t(b))$  and the proof is complete.  $\square$

Suppose  $\phi = \{\phi_t : t \geq 0\}$ ,  $\psi = \{\psi_t : t \geq 0\}$  are CP semigroups on  $\mathcal{B}(H)$  and  $\mathcal{B}(K)$  respectively. Also suppose  $\eta = \{\eta_t : t \geq 0\}$  is a family of bounded operators on  $B(K, H)$  such that  $\tau = \{(\tau_t) : t \geq 0\}$  defined



by

$$\tau_t \left( \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \right) = \begin{pmatrix} \phi_t(X) & \eta_t(Y) \\ \eta_t(Z^*)^* & \psi_t(W) \end{pmatrix}$$

is a CP semigroup on  $\mathcal{B}(H \oplus K)$ . In particular, this means that  $\eta = \{\eta_t : t \geq 0\}$  is a semigroup of bounded maps on  $\mathcal{B}(K, H)$ .

Let  $(\pi_t, V_t, G_t)$  be the minimal Stinespring dilation of  $\tau_t$ . Then clearly restrictions of  $\pi_t$  to  $\mathcal{B}(H)$  and  $\mathcal{B}(K)$  are dilations of  $\phi_t$  and  $\psi_t$ . Suppose  $\hat{H}_t = \overline{\text{span}}\{\pi_t(X)V_th : X \in \mathcal{B}(H), h \in H\}$  and  $\hat{K}_t = \overline{\text{span}}\{\pi_t(Y)V_tk : Y \in \mathcal{B}(K), k \in K\}$ . Then  $(\pi_t, V_t, \hat{H}_t)$  and  $(\pi_t, V_t, \hat{K}_t)$  are the minimal dilations of  $\phi_t$  and  $\psi_t$  respectively. Fix unit vectors  $a \in H$  and  $b \in K$  and  $\begin{pmatrix} \frac{a}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \end{pmatrix} \in H \oplus K$ . Also fix orthonormal bases  $\{e_i\}_{i \in I}$  of  $H$ ,  $\{f_j\}_{j \in J}$  of  $K$ , where  $I, J$  are some indexing sets. Then of course  $\{e_i\}_{i \in I} \cup \{f_j\}_{j \in J}$  is an orthonormal basis of  $H \oplus K$ . Define inclusion systems  $(E, \beta)$ ,  $(F, \gamma)$ , and  $(G, \delta)$  corresponding to  $\phi$ ,  $\psi$ , and  $\tau$ . Define  $D_t : F_t \rightarrow E_t$  by  $D_t = P_{E_t} \pi_t(|a\rangle\langle b|)|_{F_t}$  (where  $P_{E_t}$  is the projection onto  $E_t$ ).

**Theorem 19.** *Let  $\phi, \psi, \tau$  be CP semigroups and  $(E, \beta), (F, \gamma), (G, \delta)$  be their corresponding inclusion systems as above. Then  $D = \{D_t : t > 0\}$  is a contractive morphism from  $(F, \gamma)$  to  $(E, \beta)$ . Moreover,  $(G, \delta)$  is isomorphic to amalgamated sum of  $(E, \beta)$  and  $(F, \gamma)$  via  $D$ .*

Proof: Clearly each  $D_t$  is contractive. To see that they form a morphism, we make the following computations:

$$\begin{aligned} & \langle \beta_{s,t} \pi_{s+t}(|a\rangle\langle g|) V_{s+t} h, (D_s \otimes D_t) \gamma_{s+t} \pi_{s+t}(|b\rangle\langle g'|) V_{s+t} h' \rangle \\ &= \sum_{i,j} \langle \pi_s(|a\rangle\langle g|) V_s e_i \otimes \pi_t(|a\rangle\langle e_i|) V_t h, (D_s \otimes D_t) \pi_s(|b\rangle\langle g'|) V_s f_j \otimes \pi_t(|b\rangle\langle f_j|) V_t h' \rangle \\ &= \sum_{i,j} \langle \pi_s(|a\rangle\langle g|) V_s e_i, P_{E_s} \pi_s(|a\rangle\langle b|) \pi_s(|b\rangle\langle g'|) V_s f_j \rangle \\ & \quad \cdot \langle \pi_t(|a\rangle\langle e_i|) V_t h, P_{E_t} \pi_t(|a\rangle\langle b|) \pi_t(|b\rangle\langle f_j|) V_t h' \rangle \\ &= \sum_{i,j} \langle e_i, \eta_s(|g\rangle\langle g'|) f_j \rangle \langle h, \eta_t(|e_i\rangle\langle f_j|) h' \rangle \\ &= \langle h, \eta_t([\sum_{i,j} \langle e_i, \eta_s(|g\rangle\langle g'|) f_j \rangle |e_i\rangle\langle f_j|]) h' \rangle \\ &= \langle h, \eta_t(\eta_s(|g\rangle\langle g'|)) h' \rangle \\ &= \langle h, \eta_{s+t}(|g\rangle\langle g'|) h' \rangle \\ &= \langle \pi_{s+t}(|a\rangle\langle g|) V_{s+t} h, P_{E_{s+t}} \pi_{s+t}(|a\rangle\langle b|) \pi_{s+t}(|b\rangle\langle g'|) h' \rangle. \end{aligned}$$

This proves the first part. Now define  $U_s : G_s \rightarrow E_s \oplus_{D_s} F_s$  by

$$U_s \pi_s \left( \begin{pmatrix} \frac{a}{\sqrt{2}} \\ \frac{b}{\sqrt{2}} \end{pmatrix} \right) \left\langle \begin{pmatrix} g \\ g' \end{pmatrix} \right\rangle V_s \begin{pmatrix} h \\ h' \end{pmatrix} = \begin{bmatrix} \pi_s(|a\rangle\langle g|) V_s h \\ \pi_s(|b\rangle\langle g'|) V_s h' \end{bmatrix}.$$

Clearly  $U_s$  is linear and onto. Now

$$\begin{aligned}
\left\| \begin{bmatrix} \pi_s(|a\rangle\langle g|)V_s h \\ \pi_s(|b\rangle\langle g'|)V_s h' \end{bmatrix} \right\|^2 &= \langle h, \phi_s(|g\rangle\langle g|)h \rangle + \langle h', \psi_s(|g'\rangle\langle g'|)h' \rangle \\
&\quad + \langle h, \eta_s(|g\rangle\langle g'|)h' \rangle + \langle \eta_s(|g\rangle\langle g'|)h', h \rangle \\
&= \left\langle \begin{pmatrix} h \\ h' \end{pmatrix}, \begin{pmatrix} \phi_s(|g\rangle\langle g'|) & \eta_s(|g\rangle\langle g'|) \\ \eta_s(|g'\rangle\langle g|)^* & \psi_s(|g'\rangle\langle g'|) \end{pmatrix} \begin{pmatrix} h \\ h' \end{pmatrix} \right\rangle \\
&= \left\| \pi_s \left( \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right) \right\|^2 \left\| V_s \begin{pmatrix} h \\ h' \end{pmatrix} \right\|^2,
\end{aligned}$$

implies that  $U_s$  is a unitary operator. In a similar way, strong morphism property follows from:

$$\begin{aligned}
&(U_s \otimes U_t) \delta_{s,t} \pi_{s+t} \left( \left| \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} g \\ g' \end{pmatrix} \right| \right) V_{s+t} \begin{pmatrix} h \\ h' \end{pmatrix} \\
&= \sum_i U_s \pi_s \left( \left| \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} g \\ g' \end{pmatrix} \right| \right) V_s \begin{pmatrix} e_i \\ 0 \end{pmatrix} \otimes U_t \pi_t \left( \left| \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} e_i \\ 0 \end{pmatrix} \right| \right) V_t \begin{pmatrix} h \\ h' \end{pmatrix} \\
&\quad + \sum_j U_s \pi_s \left( \left| \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} g \\ g' \end{pmatrix} \right| \right) V_s \begin{pmatrix} 0 \\ f_j \end{pmatrix} \otimes U_t \pi_t \left( \left| \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ f_j \end{pmatrix} \right| \right) V_t \begin{pmatrix} h \\ h' \end{pmatrix} \\
&= \sum_i \begin{bmatrix} \pi_s(|a\rangle\langle g|)V_s e_i \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \pi_t(|a\rangle\langle e_i|)V_t h \\ 0 \end{bmatrix} \\
&\quad + \sum_j \begin{bmatrix} 0 \\ \pi_t(|b\rangle\langle g'|)V_t f_j \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \pi_t(|b\rangle\langle f_j|)V_t h' \end{bmatrix} \\
&= i_{s,t} \begin{bmatrix} \sum_i \pi_s(|a\rangle\langle g|)V_s e_i \otimes \pi_t(|a\rangle\langle e_i|)V_t h \\ \sum_j \pi_s(|b\rangle\langle g'|)V_s f_j \otimes \pi_t(|a\rangle\langle f_j|)V_t h' \end{bmatrix} \\
&= i_{s,t} \begin{bmatrix} \beta_{s,t} \pi_{s+t}(|a\rangle\langle g|)V_{s+t} h \\ \gamma_{s,t} \pi_{s+t}(|b\rangle\langle g'|)V_{s+t} h' \end{bmatrix} \\
&= i_{s,t} (\beta_{s,t} \oplus_D \gamma_{s,t}) \begin{bmatrix} \pi_{s+t}(|a\rangle\langle g|)V_{s+t} h \\ \pi_{s+t}(|b\rangle\langle g'|)V_{s+t} h' \end{bmatrix} \\
&= i_{s,t} (\beta_{s,t} \oplus_D \gamma_{s,t}) U_{s+t} \pi_{s+t} \left( \left| \begin{pmatrix} a/\sqrt{2} \\ b/\sqrt{2} \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} g \\ g' \end{pmatrix} \right| \right) V_{s+t} \begin{pmatrix} h \\ h' \end{pmatrix}.
\end{aligned}$$

□

Now we look at units of amalgamated products of inclusion systems.

**Lemma 20.** *Let  $(G, \gamma)$  be the amalgamated product of two inclusion systems  $(E, \beta)$  and  $(F, \gamma)$  via  $D$ . Assume that  $\text{range } (\tilde{D}_t)$  is closed for every  $t > 0$ . Suppose  $\begin{bmatrix} u_t \\ v_t \end{bmatrix}_{t>0}$  is a unit of  $(G, \delta)$ . Then  $(u_s + D_s v_s)_{s>0}$  and  $(D_s^* u_s + v_s)_{s>0}$  are units, provided they are non-trivial, in  $(E, \beta)$  and  $(F, \gamma)$  respectively.*

Proof: As  $\tilde{D}_t$  is closed, every vector of  $G_t$  is given by  $\begin{bmatrix} a \\ b \end{bmatrix}$  for some  $a \in E_t$  and  $b \in F_t$ . Let  $\begin{bmatrix} u_t \\ v_t \end{bmatrix}_{t>0}$  be a unit of the inclusion system  $(G_t = E_t \oplus_D F_t, \delta_{s,t})$ . So

$$\begin{bmatrix} u_{s+t} \\ v_{s+t} \end{bmatrix} = \delta_{s,t}^* \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix} = (\beta_{s,t}^* \oplus_D \gamma_{s,t}^*) i_{s,t}^* \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix}.$$

Suppose  $i_{s,t}^* \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} z \\ w \end{bmatrix}$ , then we claim that

$$\begin{bmatrix} I & D_s \otimes D_t \\ D_s^* \otimes D_t^* & I \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} (u_s + D_s v_s) \otimes (u_t + D_t v_t) \\ (D_s^* u_s + v_s) \otimes (D_t^* u_t + v_t) \end{pmatrix}.$$

This follows, as for  $a \otimes a' \in E_s \otimes E_t$  and  $b \otimes b' \in F_s \otimes F_t$ ,

$$\begin{aligned} & \left\langle \begin{bmatrix} I & D_s \otimes D_t \\ D_s^* \otimes D_t^* & I \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} a \otimes a' \\ b \otimes b' \end{pmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} a \otimes a' \\ b \otimes b' \end{bmatrix} \right\rangle \\ &= \left\langle i_{s,t}^* \left( \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix} \right), \begin{bmatrix} a \otimes a' \\ b \otimes b' \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix}, i_{s,t} \begin{bmatrix} a \otimes a' \\ b \otimes b' \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \left( \begin{bmatrix} a \\ 0 \end{bmatrix} \otimes \begin{bmatrix} a' \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} \otimes \begin{bmatrix} 0 \\ b' \end{bmatrix} \right) \right\rangle \\ &= \left\langle \begin{pmatrix} (u_s + D_s v_s) \otimes (u_t + D_t v_t) \\ (D_s^* u_s + v_s) \otimes (D_t^* u_t + v_t) \end{pmatrix}, \begin{pmatrix} a \otimes a' \\ b \otimes b' \end{pmatrix} \right\rangle. \end{aligned}$$

Now for  $c \in E_{s+t}, d \in F_{s+t}$

$$\begin{aligned}
\left\langle \begin{pmatrix} u_{s+t} + D_{s+t}v_{s+t} \\ D_{s+t}^*u_{s+t} + v_{s+t} \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle &= \left\langle \begin{bmatrix} u_{s+t} \\ v_{s+t} \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle \\
&= \left\langle (\beta_{s,t} \oplus_D \gamma_{s,t})^* i_{s,t}^* \left( \begin{bmatrix} u_s \\ v_s \end{bmatrix} \otimes \begin{bmatrix} u_t \\ v_t \end{bmatrix} \right), \begin{bmatrix} c \\ d \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} \beta_{s,t}(c) \\ \gamma_{s,t}(d) \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} I & D_s \otimes D_t \\ D_s^* \otimes D_t^* & I \end{bmatrix} \begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} \beta_{s,t}(c) \\ \gamma_{s,t}(d) \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} (u_s + D_s v_s) \otimes (u_t + D_t v_t) \\ (D_s^* u_s + v_s) \otimes (D_t^* u_t \otimes v_t) \end{pmatrix}, \begin{pmatrix} \beta_{s,t}(c) \\ \gamma_{s,t}(d) \end{pmatrix} \right\rangle \\
&= \left\langle \begin{pmatrix} \beta_{s,t}^*((u_s + D_s v_s) \otimes (u_t + D_t v_t)) \\ \gamma_{s,t}^*((D_s^* u_s + v_s) \otimes (D_t^* u_t \otimes v_t)) \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle
\end{aligned}$$

Further  $\{(u_s + D_s v_s) : s > 0\}$  and  $\{(D_s^* u_s + v_s) : s > 0\}$  are exponentially bounded as  $\|(u_s + D_s v_s)\| \leq \left\| \begin{bmatrix} u_s \\ v_s \end{bmatrix} \right\|_{G_s}$  and similarly  $\|(D_s^* u_s + v_s)\| \leq \left\| \begin{bmatrix} u_s \\ v_s \end{bmatrix} \right\|_{G_s}$ . Therefore they are units in the corresponding inclusion systems.  $\square$

The Theorem 19 answers Powers question, even for general corners (not just those given by units), by showing that the product system of the CP semigroup formed is the amalgamation. Lemma 20 helps us in computing the units of such amalgamations. We also would like to compute Arveson index of the product systems. However, we have technical problem here. The index of a product system as defined by Arveson ([1]) needs measurability of units and we have not imposed any measurability structure on our inclusion systems or product systems. We do not intend to develop a theory of measurable inclusion systems here. Instead, we restrict ourselves to considering subsystems of measurable product systems, in the sense of Arveson.

Here after product systems we consider are Arveson systems and the units we consider are measurable.

Let  $\mathcal{U}^{\mathcal{E}}$  denote the units of a product system  $\mathcal{E}$ . Then the measurability ensures the existence a function

$$\gamma : \mathcal{U}^{\mathcal{E}} \times \mathcal{U}^{\mathcal{E}} \rightarrow \mathbb{C}$$

called the covariance function satisfying:

$$\langle u_t, v_t \rangle = e^{t\gamma(u,v)} \quad \forall t,$$

for units  $u, v$ . The function  $\gamma$  is a conditionally positive definite function [1]. If  $Z$  is a non-empty subset of  $\mathcal{U}^{\mathcal{E}}$ , we may do the usual GNS construction for the kernel  $\gamma$  restricted to  $Z \times Z$  to obtain a Hilbert

space  $H_Z$ , which we call as the Arveson Hilbert space associated to  $Z$ . Note that the index of the product system  $\mathcal{E}$  is nothing but the dimension of  $\mathcal{K} := H_{\mathcal{U}^\mathcal{E}}$  (Arveson Hilbert space of  $\mathcal{U}^\mathcal{E}$ ). In [1], it is shown that there exists a bijection  $u \mapsto (\lambda(u), \mu(u)) \in \mathbb{C} \times \mathcal{K}$ , between  $\mathcal{U}^\mathcal{E}$  and  $\mathbb{C} \times \mathcal{K}$ , satisfying

$$\gamma(u, u') = \overline{\lambda(u)} + \lambda(u') + \langle \mu(u), \mu(u') \rangle.$$

In the following, for simplicity of notation, though we have different product systems, we will be using same  $\lambda$  and  $\mu$  for the corresponding bijections. This shouldn't cause any confusion. We need couple of lemmas before we state our main theorem. We omit the proof of the first Lemma.

**Lemma 21.** *Let  $\gamma$  be the covariance kernel on the set of all units in a product system  $\mathcal{E}$ . Suppose there is a function  $a : \mathcal{U}^\mathcal{E} \rightarrow \mathbb{C}$  such that the function  $L : \mathcal{U}^\mathcal{E} \times \mathcal{U}^\mathcal{E} \rightarrow \mathbb{C}$  defined by  $L(x, y) = \gamma(x, y) - \overline{a(x)} - a(y)$ ,  $x, y \in \mathcal{U}^\mathcal{E}$  is positive definite, then*

$$\sup\{ \text{rank}[L(x_i, x_j)]_{n \times n} : x_i, x_j \in \mathcal{U}^\mathcal{E}, n \geq 1 \} = \text{index}(\mathcal{E})$$

Recall that units of a spatial product system generate a type I product subsystem. The index of the product system is same as the index of this subsystem. Further, any type I product system is isomorphic to the exponential product system ([1]) or the product system consisting of symmetric Fock spaces  $\{\Gamma(L^2[0, t], K)\}_{t>0}$  where  $K$  is a Hilbert space with  $\dim K$  equal to the index of the product system. In this picture of type I product system, units are parametrized by exponential vectors:

$$\{e^{qt}e(x\chi|_t)\}_{t>0} \quad (q, x) \in \mathbb{C} \times K.$$

The automorphisms of this product system is parametrized by triples  $\phi := [q, z, U]$ , where  $q \in \mathbb{R}$ ,  $z \in K$ ,  $U$  is a unitary in  $B(K)$ , and  $\phi$  acts on the exponential vectors by

$$\phi e(x\chi|_t) = e^{-iqt - \frac{\|z\|^2 t}{2} - \langle z, Ux \rangle t} e((z + Ux)\chi|_t).$$

Then the adjoint of  $\phi$ ,  $\phi^*$  is parameterized by the tuple  $[-q, -U^*z, U^*]$ .

**Lemma 22.** *Let  $A$  be a non-empty subset of a separable Hilbert space  $K$ . Then the set of all units in the product subsystem  $\mathcal{E}$  of  $\Gamma(L^2[0, t], K)$ , generated by units  $\{e(x\chi|_t) : x \in A\}_{t>0}$  is given by*

$$\{e^{ct}e(y + x_0)\chi|_t) : y \in \overline{\text{span}}(A - x_0), c \in \mathbb{C}\}_{t>0}$$

where  $x_0$  is any fixed vector in  $A$ . In particular

$$\text{index}(\mathcal{E}) = \dim \overline{\text{span}}(A - x_0)$$

Proof: Fix  $x_0 \in A$ . As the type I product system is transitive, we can get an automorphism  $\phi = [q, U, x_0]$  which sends vacuum unit to  $e^{-iqt - \frac{\|x_0\|^2 t}{2}} e(x_0\chi|_t)$ . Then we claim that it is enough to show the following assertion : For a subset  $B \subseteq K$ , The set  $\{e(y\chi|_t) : y \in B \cup 0\}$  generates units of the form

$$\{e^{\alpha t}e(y\chi|_t) : y \in \overline{\text{span}} B, \alpha \in \mathbb{C}\}.$$

First assume that the assertion is true. For  $x \in A$ ,  $\phi^*e(x\chi|_t) = e^{igt - \frac{\|x_0\|^2 t}{2} + \langle x_0, x \rangle t} e((U^*(x - x_0))\chi|_t)$ . Set  $C = \{U^*(x - x_0) : x \in A\}$ . Under  $\phi^*$ , The set  $\{e(x\chi|_t) : x \in A\}$  maps to  $\{e^{igt - \frac{\|x_0\|^2 t}{2} + \langle x_0, x \rangle t} e(x\chi|_t) : x \in C\}$ . As  $C = C \cup \{0\}$ , by the assertion we get that the set  $\{e(x\chi|_t) : x \in C\}$  generates units of the form

$$\{e^{\alpha t} e(x\chi|_t) : x \in \overline{\text{span}} C, \alpha \in \mathbb{C}\}.$$

Now under the image of  $\phi$ , we get that the set  $\{e(x\chi|_t) : x \in A\}$  generates units of the form

$$\{e^{\alpha t} e(x\chi|_t) : x \in \overline{\text{span}}(UC) + x_0, \alpha \in \mathbb{C}\}.$$

Now  $\{x : x \in \overline{\text{span}}(UC) + x_0\} = \{y + x_0 : y \in \overline{\text{span}}(A - x_0)\}$ . Hence the lemma is proved. Now we will prove the assertion. Let  $K_1 = \overline{\text{span}}B$ . Now in the subsystem  $\Gamma_{\text{sym}}(L^2([0, t], K_1))$ , consider the set  $\{e(z\chi|_t) : z \in B \cup \{0\}\}$ . They generate exponentials of all step functions taking values in  $B \cup \{0\}$ . As  $B$  is total in  $K_1$ . So by a result of Skeide [11], it is all of  $\Gamma_{\text{sym}}(L^2([0, t], K_1))$ . Hence its units are parameterized by  $\{e^{\alpha t} e(y\chi|_t) : y \in K_1, \alpha \in \mathbb{C}\}$ .

Let us denote the product system generated by the exponential vectors  $\{e(x\chi|_t) : x \in A\}$  by  $\mathcal{F}$ . We wish to calculate its index. Covariance function is the restriction of the covariance function of  $\Gamma_{\text{sym}}(L^2([0, t], K_1))$ , which is  $\gamma((\alpha, x), (\beta, y)) = \bar{\alpha} + \beta + \langle x, y \rangle$ . Units of  $\mathcal{F}$  are parameterized by  $\mathbb{C} \times [x_0 + \overline{\text{span}}(A - x_0)]$ . Under the automorphism map  $\phi^*$ ,  $[x_0 + \overline{\text{span}}(A - x_0)]$  maps to the subspace  $U^*(\overline{\text{span}}(A - x_0))$ . So we get  $\text{index}(\mathcal{F}) = \dim U^*(\overline{\text{span}}(A - x_0)) = \dim(\overline{\text{span}}(A - x_0))$ .  $\square$

**Lemma 23.** *Let  $\mathcal{E}$  be a spatial product system and  $Z \subset \mathcal{U}^\mathcal{E}$  be a subset of the set of all units in  $\mathcal{E}$ . Let  $H_Z$  be the Arveson Hilbert space associated to  $Z$ . Then  $\dim H_Z = \text{ind } \mathcal{E}$  if and only if  $\overline{\text{span}}\{u_{t_1}^1 \otimes u_{t_2}^2 \otimes \cdots \otimes u_{t_k}^k : 1 \leq i \leq k, u^i \in Z, \sum t_j = t, k \geq 1\} = \mathcal{E}_t$  for all  $t > 0$ .*

*Proof:* With out loss of generality, we may assume that the given product system is of type I. We then identify the product system with symmetric Fock space product system. Then  $Z$  can be identified with a subset of  $\mathbb{C} \times H_{\mathcal{U}^\mathcal{E}}$ . Take  $A = \{x : (\alpha, x) \in Z \text{ for some } \alpha \in \mathbb{C}\}$  from the construction of Arveson Hilbert space  $H_Z$ , it follows easily that  $H_Z = \overline{\text{span}}(A - x_0)$ , where  $x_0$  is any fixed vector in  $A$ . Now the result follows from Lemma 22.  $\square$

**Theorem 24.** *Suppose  $\phi = \{\phi_t : t \geq 0\}$  and  $\psi = \{\psi_t : t \geq 0\}$  are two  $E_0$  semigroups on  $\mathcal{B}(\mathcal{H})$  and  $\mathcal{B}(\mathcal{K})$  respectively and  $U = \{U_t : t \geq 0\}$  and  $V = \{V_t : t \geq 0\}$  are two strongly continuous semigroups of contractions which intertwine  $\phi_t$  and  $\psi_t$  respectively. Consider the CP semigroup  $\tau_t$  on  $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$  defined by  $\tau_t \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} \phi_t(X) & U_t Y V_t^* \\ V_t Z U_t^* & \psi_t(W) \end{pmatrix}$ . Let  $(\mathcal{E}, B)$ ,  $(\mathcal{F}, C)$  and  $(\mathcal{G}, W)$  be the Arveson's product systems associated to  $\phi$ ,  $\psi$  and  $\tau$ . Then the following holds:*

- (1) *there exist two units  $(u^0)_{t>0}$  and  $(v^0)_{t>0}$  of  $(\mathcal{E}, B)$  and  $(\mathcal{F}, C)$  respectively such that  $D := \{D_t = |u_t^0\rangle\langle v_t^0| : t > 0\}$  from  $\mathcal{F}$  to  $\mathcal{E}$  such that  $\mathcal{E} \otimes_D \mathcal{F} = \mathcal{G}$ .*

(2) The type I part of the amalgamated product is the amalgamated product of type I parts of  $\mathcal{E}$  and

$$\mathcal{F}: (\mathcal{E}^I \otimes_D \mathcal{F}^I) = (\mathcal{E} \otimes_D \mathcal{F})^I;$$

(3)

$$\text{index}(\mathcal{E} \otimes_D \mathcal{F}) = \begin{cases} \text{index}(\mathcal{E}) + \text{index}(\mathcal{F}) & \text{if } \|u_t^0\| = \|v_t^0\| = 1 \forall t > 0; \\ \text{index}(\mathcal{E}) + \text{index}(\mathcal{F}) + 1 & \text{otherwise.} \end{cases}$$

Proof: Strong continuity properties of  $U$  and  $V$  imply that the product system associated to the  $E_0$  dilation of  $\tau_t$  is an Arveson's product system. Let  $(G, \delta)$  be the inclusion system of  $\tau$ . Now from Theorem 19,  $G_t = \mathcal{E}_t \oplus_{D_t} \mathcal{F}_t$ . We conclude that  $\mathcal{G} = \mathcal{E} \otimes_D \mathcal{F}$ , where

$$D_t = P_{\mathcal{E}_t} \pi_t(|a\rangle\langle b|)|_{\mathcal{F}_t} = |U_t a\rangle\langle V_t b|,$$

(Here  $\pi_t$  denotes the minimal dilation of  $\tau_t$ .) Take  $u_t^0 = U_t a$  and  $v_t^0 = V_t b$ . Then clearly  $(u_t^0)_{t>0}$  and  $(v_t^0)_{t>0}$  are units of  $(\mathcal{E}, B)$  and  $(\mathcal{F}, C)$  respectively.

Consider  $M_t = \overline{\text{span}}\left\{ \begin{bmatrix} u_t \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ v_t \end{bmatrix}, u \in \mathcal{U}^{\mathcal{E}}, v \in \mathcal{U}^{\mathcal{F}} \right\}$ . As  $u \in \mathcal{U}^{\mathcal{E}}$  and  $v \in \mathcal{U}^{\mathcal{F}}$ ,  $\begin{bmatrix} u \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ v \end{bmatrix}$  are strong units in  $(G, \delta)$ , and  $(M, \delta|_M)$  is an inclusion subsystem of  $(G, \delta)$ . Let  $(\mathcal{M}, W|_{\mathcal{M}})$  be its generated product system, we get that  $(\mathcal{M}, W|_{\mathcal{M}}) \supset \mathcal{E}^I \otimes_D \mathcal{F}^I$ . As  $(\mathcal{M}, W|_{\mathcal{M}})$  is a product subsystem of  $(\mathcal{G}^I, W)$ , we have  $\mathcal{E}^I \otimes_D \mathcal{F}^I$  as a product subsystem of  $(\mathcal{G}^I, W)$ . In particular amalgamation of two type I system is again type I. Now let  $\begin{bmatrix} u \\ v \end{bmatrix}$  be a unit of  $(G, \delta)$ .  $D_s = |u_s^0\rangle\langle v_s^0|$ , implies  $\tilde{D}_t$  is closed. So invoking the

Lemma 20 we can conclude that  $\begin{bmatrix} u_t \\ v_t \end{bmatrix} \in M_t$  for every  $t > 0$ . In particular it implies that  $(\mathcal{G}^I, W)$  is a product subsystem of  $(\mathcal{M}, W|_{\mathcal{M}})$ . So  $(\mathcal{E}^I \otimes_D \mathcal{F}^I) \simeq \mathcal{G}^I$ , proving (2).

So from the argument above, it follows that the set

$$Z = \left\{ \widehat{\begin{bmatrix} u \\ 0 \end{bmatrix}} : u \in \mathcal{U}^{\mathcal{E}} \right\} \cup \left\{ \widehat{\begin{bmatrix} 0 \\ v \end{bmatrix}} : v \in \mathcal{U}^{\mathcal{F}} \right\} \subset \mathcal{U}^{\mathcal{G}}$$

generates  $\mathcal{G}^I$  (Here the lift to the generated product system of a unit  $x$  of an inclusion system is denoted by  $\hat{x}$ .) So from 23,  $\text{ind}(\mathcal{G}) = \dim H_Z$ . So it is enough to calculate the rank of the covariance kernel on  $Z$ . The covariance function can be computed as follows. For arbitrary units  $u \in \mathcal{U}^{\mathcal{E}}, v \in \mathcal{U}^{\mathcal{F}}$ ,

$$\left\langle \widehat{\begin{bmatrix} u_t \\ 0 \end{bmatrix}}, \widehat{\begin{bmatrix} 0 \\ v_t \end{bmatrix}} \right\rangle = \langle u_t, |u_t^0\rangle\langle v_t^0|v_t \rangle = e^{t(\gamma(u, u^0) + \gamma(v^0, v))}.$$

Therefore,

$$\gamma\left(\widehat{\begin{bmatrix} u \\ 0 \end{bmatrix}}, \widehat{\begin{bmatrix} 0 \\ v \end{bmatrix}}\right) = \overline{\lambda(u)} + \lambda(u^0) + \langle \mu(u), \mu(u^0) \rangle + \overline{\lambda(v)} + \lambda(v) + \langle \mu(v^0), \mu(v) \rangle.$$

Similarly, for  $u, u' \in \mathcal{U}^\mathcal{E}$  and  $v, v' \in \mathcal{U}^\mathcal{F}$ ,

$$\begin{aligned}\gamma\left(\widehat{\begin{bmatrix} u \\ 0 \end{bmatrix}}, \widehat{\begin{bmatrix} u' \\ 0 \end{bmatrix}}\right) &= \overline{\lambda(u)} + \lambda(u') + \langle \mu(u), \mu(u') \rangle, \\ \gamma\left(\widehat{\begin{bmatrix} u \\ 0 \end{bmatrix}}, \widehat{\begin{bmatrix} 0 \\ v \end{bmatrix}}\right) &= \overline{\lambda(v)} + \lambda(v') + \langle \mu(v), \mu(v') \rangle.\end{aligned}$$

Now take  $Y = \mathcal{U}^\mathcal{E} \cup \mathcal{U}^\mathcal{F}$  and define  $a : Y \rightarrow \mathbb{C}$  by

$$a(u) = \lambda(u) + \langle \mu(u^0), \mu(u) - \frac{1}{2}\mu(u^0) \rangle \text{ if } u \in \mathcal{U}^\mathcal{E}$$

and

$$a(v) = \lambda(u^0) + \frac{1}{2}\langle \mu(u^0), \mu(u^0) \rangle + \overline{\lambda(v^0)} + \lambda(v) + \langle \mu(v^0), \mu(v) \rangle \text{ if } v \in \mathcal{U}^\mathcal{F}.$$

Similarly define  $L : Y \times Y \rightarrow \mathbb{C}$  by

$$L(x, y) = \gamma(\widehat{[x]}, \widehat{[y]}) - \overline{a(x)} - a(y).$$

(Here  $[x]$  denotes  $\begin{bmatrix} x \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ x \end{bmatrix}$  according as  $x$  is in  $\mathcal{U}^\mathcal{E}$  or  $\mathcal{U}^\mathcal{F}$ .) Then by direct computation: For  $u, u' \in \mathcal{U}^\mathcal{E}$  and  $v, v' \in \mathcal{U}^\mathcal{F}$ ,

$$\begin{aligned}L(u, u') &= \langle \mu(u) - \mu(u^0), \mu(u') - \mu(u^0) \rangle \\ L(u, v) &= 0 \\ L(v, v') &= \langle \mu(v) - \mu(v^0), \mu(v') - \mu(v^0) \rangle + p\end{aligned}$$

where  $p := -[\overline{\lambda(u^0)} + \lambda(u^0) + \overline{\lambda(v^0)} + \lambda(v^0) + \langle \mu(u^0), \mu(u^0) \rangle + \langle \mu(v^0), \mu(v^0) \rangle] = -[\gamma(u^0, u^0) + \gamma(v^0, v^0)]$ .

It is to be noted that as  $e^{-tp} = \|u_t^0\| \cdot \|v_t^0\| \leq 1$  for all  $t$ ,  $p$  is non-negative and  $p = 0$  iff  $\|u_t^0\| = \|v_t^0\| = 1$  for all  $t$ . So taking direct sum of the range space of  $\mu$  with  $\mathbb{C}$ , we get

$$L(v, v') = \langle (\mu(v) - \mu(v^0)) \oplus \sqrt{p}, (\mu(v') - \mu(v^0)) \oplus \sqrt{p} \rangle.$$

For any unit  $u \in \mathcal{U}^\mathcal{E}$ , we can find another unit  $\tilde{u} \in \mathcal{U}^\mathcal{E}$  such that  $\mu(u) = \mu(\tilde{u}) - \mu(u^0)$ . Then it is clear that maximal rank of  $[L(x_i, x_j)]$  with  $x_1, \dots, x_n \in \mathcal{U}^\mathcal{E}$  is equal to index  $(\mathcal{E})$ . Similarly, maximal rank of  $[L(y_i, y_j)]$  with  $y_1, \dots, y_n \in \mathcal{U}^\mathcal{F}$  is equal to index  $(\mathcal{F}) + 1$ , if  $p > 0$  and is equal to index  $(\mathcal{F})$  if  $p = 0$ . The theorem follows from the Lemma 21.  $\square$

**Acknowledgements:** B. V. Rajarama Bhat thanks UKIERI for financial support.



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